The Order Up-To Inventory Model

- Inventory Control
- Best Order Up-To Level
- Best Service Level
- Impact of Lead Time

1. Medtronic’s InSync Pacemaker

- Model 7272. Implanted in a patient after a cardiac surgery.
- One distribution center (DC) in Mounds View, Minnesota.
- About 500 sales territories. Majority of FGI is held by sales representatives.
- Consider Susan Magnotto’s territory in Madison, Wisconsin.
**Demand and Inventory at the DC**

Avg. monthly demand = 349

Std. Dev. of monthly demand = 122.28

Avg. weekly demand = 349/4.33 = 80.6

Standard deviation of weekly demand = 122.38/√4.33 = 58.81

(Assume 4.33 weeks per month and independent weekly demands.)

**Demand and Inventory in Susan’s Territory**

Total annual demand = 75

Average daily demand = 0.29 units (75/260), assuming 5 days per week.

Poisson demand distribution works better for slow moving items.
Medtronic’s Inventory Problem

- Patients and surgeons do not tolerate backorders.
- The pacemaker is small and has a long shelf life.
- Sales incentive system.
- Each representative is given a par level which is set quarterly based on previous sales and anticipated demand.

Objective: Because the gross margins are high, Medtronic wants an inventory control policy to minimize inventory investment while maintaining a very high fill rate.

Review: Reasons to Hold Inventory

- Pipeline Inventory
- Seasonal Inventory
- Cycle Inventory
- Decoupling Inventory/Buffers
- Safety Inventory
Review: Reasons to Hold Less Inventory

- Inventory might become obsolete.
- Inventory might perish.
- Inventory might disappear.
- Inventory requires storage space and other overhead cost.
- Opportunity cost.

Review: Inventory Costs

- Holding or Carrying cost  
  storage cost: facility, handling  
  risk cost: depreciation, pilferage, insurance  
  opportunity cost  

- Ordering or Setup cost  
  cost placing an order or changing machine setups

- Shortage costs or Lost Sales  
  costs of canceling an order or penalty

  Annual cost ≈ 20% to 40% of the inventory’s worth
Review: Inventory Performance

- **Throughput rate** = average daily sales
- **Throughput time** = days of supply
- **avg. Inventory value** = avg. daily sales * avg. throughput time
- **Days of supply** = \( \frac{\text{average inventory value}}{\text{average daily sales}} \)

Review: Inventory Performance

- **Monthly Inventory turn** = \( \frac{\text{Cost of Goods Sold in one month}}{\text{average inventory value}} \)
- **Service level** = in-stock probability before the replenishment order arrives
- **Fill rate** = \( \frac{\text{number of sales}}{\text{number of demands}} \)
Review: Multi-Period Inventory Models

Q model: fixed order quantity

\[ R \] is the reorder point and is based on lead time \( L \) and the forecast. Place a new order whenever the inventory level drops to \( R \).

Review: P model: fixed time period

\[ T \] is the review period.
\[ S \] is the target inventory level determined by the forecasts.

We place an order to bring the inventory level up to \( S \).
Review: Safety Stock

- amount of inventory carried in addition to the expected demand, in order to avoid shortages when demand increases

Service level = probability of no shortage

\[ = P(\text{demand} \leq \text{inventory}) \]
\[ = P(\text{demand} \leq \text{E}(D) + \text{safety stock}) \]

- depends on service level, demand variability, order lead time
- service level depends on Holding cost \(\leftrightarrow\) Shortage cost

Review: Q Models with Safety Stock

Timespan = Lead time \(L\) (in days)

\[ R = \text{expected demand during } L + \text{safety stock} \]
\[ \approx \bar{d} \cdot L + z \cdot \sqrt{L} \cdot \sigma_d \]

\( \bar{d} \) = daily demand
\( \sigma_d \) = std dev. of daily demand

Service level or probability of no shortage

\[ = 95\% (99\%) \Rightarrow z = 1.64 (2.33) \]
Review: P Models with Safety Stock

Timespan = length of review period + lead time = $T + L$

**Target Inventory** = expected demand + safety stock

$$= \bar{d}(T + L) + \sigma \cdot \sqrt{T + L} \cdot \sigma$$

Order Quantity = target inventory – inventory position

2. The Order Up-To Model (P models)

- Time is divided into periods of equal length, e.g., one week.
- During a period the following sequence of events occurs:
  - A replenishment order can be submitted.
  - A previous order is received. (lead times = $l$)
  - Random demand occurs.
Order Up-To Model Definitions

- **On-order inventory (pipeline inventory)** = the number of units that have been ordered but have not been received.
- **On-hand inventory** = number of units physically in stock
- **Backorder** = total amount of demand yet to be satisfied.
- **Inventory level** = On-hand inventory - Backorder.
- **Inventory position** = On-order inventory + Inventory level.
- **Order up-to level, S** is the maximum inventory position or target inventory level or **base stock level**.

Order Up-To Model Implementation

*Each period’s order quantity = S – Inventory position*

- Suppose \( S = 4 \).
  - If begins with an inventory position = 1, order \( 4 - 1 = 3 \)
  - If begins with an inventory position = -3, order \( 4 - (-3) = 7 \)

- \( S = 4 \). We begin with an inventory position = 1 and order 3.
  - If demand were \( 10 \) in period 1, then the inventory position at the start of period 2 is \( 1 - 10 + 3 = -6 \). \( \Rightarrow \) order \( 4 - (-6) = 10 \) units

*Pull system*: order quantity = the previous period’s demand
Solving the Order-up-to Model

- Given an order-up-to level $S$
  - What is the average inventory?
  - What is the expected lost sale?

- What is the best order-up-to level?

Order Up-To Level and Inventory Level

- On-order inventory + Inventory level at the beginning of Period 1 = $S$
- Inventory level at the end of Period $l + 1 = S - \text{demand over recent } l + 1 \text{ periods}$.
- Ex: $S = 6$, $l = 3$, and 2 units on-hand at the start of period 1

![Diagram showing order up-to level and inventory level over time]

Inventory level at the end of period 4

$$= 6 - D_1 - D_2 - D_3 - D_4$$
Similarity with a Newsvendor Model

This is like a Newsvendor model in which the order quantity is $S$ and the demand distribution is demand over $l+1$ periods.

Expected On-Hand Inventory and Backorder

- **Expected on-hand inventory** at the end of a period can be evaluated like *Expected left over inventory* in the Newsvendor model with $Q = S$.
- **Expected backorder** at the end of a period can be evaluated like *Expected lost sales* in the Newsvendor model with $Q = S$.
- **Expected on-order inventory**
  
  $= \text{Expected demand in a period} \times \text{lead time}$

  This comes from Little’s Law. Note that it equals the expected demand over $l$ periods, not $l+1$ periods.
Key Performance Measures

- The **stockout probability** is the probability at least one unit is backordered in a period:
  \[
  \text{Stockout probability} = \text{Prob}[\text{Demand over (l + 1) periods} > S]
  = 1 - \text{Prob}[\text{Demand over (l + 1) periods} \leq S]
  \]

- The **in-stock probability** is the probability all demand is filled in a period:
  \[
  \text{In-stock probability} = 1 - \text{Stockout probability}
  = \text{Prob}[\text{Demand over (l + 1) periods} \leq S]
  \]

- The **fill rate** is the fraction of demand within a period that is NOT backordered:
  \[
  \text{Fill rate} = 1 - \frac{\text{Expected backorder}}{\text{Expected demand in one period}}
  \]

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**Medtronic: Demand over l+1 Periods at DC**

**DC**

- The period length is one week, the replenishment lead time is three weeks, \( l = 3 \)

- Assume demand is normally distributed:
  - Mean weekly demand is 80.6 (from demand data)
  - Standard deviation of weekly demand is 58.81
  - Expected demand over \( l + 1 \) weeks is \((3 + 1) \times 80.6 = 322.4\)
  - Standard deviation of demand over \( l + 1 \) weeks is 117.6

  \[
  \sqrt{3 + 1} \times 58.81 = 117.6
  \]
DC’s Expected Backorder Assuming $S = 625$

*Expected backorder* $\approx$ *Expected lost sales* in a Newsvendor model:

- Suppose $S = 625$ at the DC
- Normalize the order up-to level: $z = \frac{S - \mu}{\sigma} = \frac{625 - 322.4}{117.6} = 2.57$

- Lookup $L(z)$ in the Standard Normal Loss Function Table: $L(2.57)=0.0016$
- Convert expected lost sales, $L(z)$, into the expected backorder with the actual normal distribution that represents demand over $(l+1)$ periods:

$$\text{Expected backorder} = \sigma \times L(z) = 117.6 \times 0.0016 = 0.19$$

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Other DC Performance Measures

- % of demand is filled immediately (not backorders)

$$\text{Fill rate} = 1 - \frac{\text{Expected backorder}}{\text{Expected demand in one period}} = 1 - \frac{0.19}{80.6} = 99.76\%.$$  

- Average number of units on-hand at the end of a period.

$$\text{Expected on-hand inventory} = S - \text{Expected demand over } (l+1) \text{ periods} + \text{Expected backorder} = 625 - 322.4 + 0.19 = 302.8.$$ 

- There are 241.8 units on-order at any given time.

$$\text{Expected on-order inventory} = \text{Expected demand in one period} \times \text{Lead time} = 80.6 \times 3 = 241.8.$$
Choose $S$ with Normally Distributed Demand

Suppose the target in-stock probability at the DC is $99.9\%$

- From the Standard Normal Distribution Function Table, $\Phi(3.08)=0.9990$
- So we choose $z = 3.08$
- To convert $z$ into an order up-to level:

$$S = \mu + z \times \sigma = 322.4 + 3.08 \times 117.6$$
$$= 685$$

- Note that $\mu$ and $\sigma$ are the parameters of the normal distribution that describes demand over $l + 1$ periods.

Medtronic: Demand at Susan’s Region

Susan’s territory:

- The period length is one day, the replenishment lead time is one day, $l=1$
- Assume demand is Poisson distributed:
  - Mean daily demand is 0.29 (from demand data)
  - Expected demand over $l+1$ days is $2 \times 0.29 = 0.58$
  - Recall, the Poisson is completely defined by its mean (standard deviation is the square root of the mean)
Using EXCEL to Calculate the Poisson Loss

<table>
<thead>
<tr>
<th>Poisson Mean demand</th>
<th>0.58 lost sales prob</th>
<th>Order Size probability</th>
<th>3 expected loss</th>
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Suppose $S = 3$

- Expected backorder $L(S) = 0.00335$
- Stock out probability = 0.0029

Performance Measures in Susan’s Territory

- Order-up-to level $S = 3$
  - Expected backorder = $L(S) = 0.00335$
  - In-stock = 99.70%
  - Fill rate = $1 - 0.00335 / 0.29 = 98.84$
  - Expected on-hand = $S - \text{demand over t+1 periods} + \text{backorder} = 3 - 0.58 + 0.00335 = 2.42$
  - Expected on-order inventory = Demand over the lead time = 0.29
Choose \( S \) with Poisson Demand

- Period length is 1 day, replenishment lead time is \( l = 1 \)
- Demand over \( l+1 \) days is Poisson with mean 0.58
- Target in-stock is \( 99.9\% \)
- In Susan’s territory, \( S = 4 \) is the smallest value that meets the target in-stock probability:

<table>
<thead>
<tr>
<th>( S )</th>
<th>Probability { Demand over ( l+1 ) periods ( \leq S ) )</th>
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Best Order-up-to Level via Cost Minimization

- If \( S \) is too high, there are holding costs, \( C_h = 0.000337 \times p \)
- If \( S \) is too low, there are lost sales, \( C_u = 0.75 \times p \)
- Best order up-to level must satisfies

\[
P(\text{Demand over } l + 1 \text{ periods } \leq S) = \frac{C_u}{C_h + C_u} = \frac{0.75p}{0.000337p + 0.75p} = 0.9996
\]

- Optimal in-stock probability is 99.96% because

\[
\text{In stock probability} = P(\text{Demand over } l + 1 \text{ periods } \leq S) = 0.9996
\]
14.8 Impact of Period Length on DC Cost

- Increasing the period length leads to larger and less frequent orders:

\[ \text{Inventory Holding Costs vs. Ordering Costs} \]

- Costs:
  - Ordering costs = $275 per order
  - Holding costs = 25% per year
  - Unit cost = $50
  - Holding cost per unit per year = 25% x $50 = 12.5

- Period length of 4 weeks minimizes costs:
  - This implies the average order quantity is 4 x 100 = 400 units

- EOQ model: 
  \[ Q = \sqrt{\frac{2 \times K \times R}{h}} = \sqrt{\frac{2 \times 275 \times 5200}{12.5}} = 478 \]
14.9 Better Service Requires More Inventory

More inventory is needed as demand uncertainty increases for any fixed fill rate.

The required inventory is more sensitive to the fill rate level as demand uncertainty increases.

Shorten Lead Times Reduce Inventory

Reducing the lead time reduces expected inventory, especially as the target fill rate increases.

The impact of lead time on expected inventory for four fill rate targets, 99.9%, 99.5%, 99.0% and 98%, top curve to bottom curve respectively.
Do Not Forget About Pipeline Inventory

Reducing the lead time reduces expected inventory and pipeline inventory.

The impact on pipeline inventory can be even more dramatic that the impact on expected inventory.

The higher the order up-to level, the better the service.

Key factors that determine the amount of inventory needed:
- The length of the replenishment lead time.
- The desired service level (fill rate or in-stock probability).
- Demand uncertainty.

When inventory obsolescence is not an issue, the optimal service level is generally quite high.