Fiber Bragg Grating Modeling

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1. Introduction

In this report, we briefly review the coupled-mode theory for fiber Bragg grating simulation, and then discuss several types of fiber Bragg grating’s design and modeling, including WDM gratings, dispersion compensation gratings and tilted gratings.

2. Coupled Mode Theory

Coupled mode theory [1,2,3] is a good tool for obtaining quantitative information about the diffraction efficiency and spectral dependence of fiber gratings. It is a straightforward and intuitive method to accurately model the optical properties of most fiber gratings of interest.

Assume the effective refractive index $n_{\text{eff}}$ of the guided modes of interest, described by:

$$
\delta n_{\text{eff}}(z) = \delta n_{\text{dc}}(z) + \delta n_{\text{ac}}(z) \cos \left[ \frac{2\pi}{\Lambda} z + \phi(z) \right]
$$

(2.1)

where $\delta n_{\text{dc}}$ is the "dc" index change over the grating length, $\delta n_{\text{ac}}$ represents the distribution of the index change due to apodization. The constant $C$ is a parameter that accounts for additional UV-induced change of the average index along the fiber. $\Lambda$ is the nominal period, and $\phi(z)$ describes grating chirp. Considered two counter propagating modes confined to the core of an optical fiber in which is formed a Bragg grating, the electric fields of the forward and backward waves in this grating can be expressed as $R(z) = A(z) \exp( j \beta z)$ and $S(z) = B(z) \exp(-j \beta z)$ respectively, where

$$
\delta = \beta - \frac{\pi}{\Lambda} = \beta - \beta_D = 2\pi \left( \frac{1}{\lambda} - \frac{1}{\lambda_D} \right)
$$

(2.2)

$\beta = (2\pi/\lambda)n_{\text{eff}}$ is the mode propagation constant. $\lambda_D = 2\lambda_{\text{eff}} \Lambda$ is the "design wavelength" for Bragg scattering by an infinitesimally weak grating ($\delta n_{\text{eff}} \to 0$) with a period $\Lambda$.

In the ideal-mode approximation to coupled-mode theory, we assume that the transverse component of the electric field can be written as a superposition of the ideal modes labeled $j$ (i.e., the modes in an ideal waveguide with no grating perturbation), such that

$$
\tilde{E}_i(x,y,z,t) = \sum_j [A_j(z) \exp(j\beta_j z) + B_j(z) \exp(-j\beta_j z)] \cdot \tilde{e}_{j}(x,y) \exp(-i\omega t)
$$

(2.3)
where $A_j(z)$ and $B_j(z)$ are slowly varying amplitude of the $j$th mode traveling in the $+z$ and $-z$ directions, respectively. The transverse mode fields $\bar{e}_x(x,y)$ might describe the bound core or radiation LP modes, or they might describe cladding modes. $\beta_j=2\pi/\lambda n_{df}$ is the mode propagation constant. While the modes are orthogonal in an ideal waveguide and hence, do not exchange energy, the presence of a dielectric perturbation causes the modes to be coupled such that the amplitudes $A_j$ and $B_j$ of the $j$th mode evolve along the $z$ axis according to

$$\frac{dA_j}{dz} = \sum_k A_k \left( K'_{jk} + K''_{jk} \right) \exp\left[ i(\beta_k - \beta_j) z \right] + \sum_k B_k \left( K'_{jk} - K''_{jk} \right) \exp\left[ -i(\beta_k + \beta_j) z \right]$$

$$\frac{dB_j}{dz} = -\sum_k A_k \left( K'_{jk} - K''_{jk} \right) \exp\left[ i(\beta_k + \beta_j) z \right] - \sum_k B_k \left( K'_{jk} + K''_{jk} \right) \exp\left[ -i(\beta_k - \beta_j) z \right]$$

where $K'_{jk}(z)$ is the transverse coupling coefficient between modes $j$ and $k$ given by

$$K'_{jk}(z) = \frac{\pi}{4} \int \int \Delta \epsilon(x,y,z) \bar{e}_x(x,y) \cdot \bar{e}_y(x,y) \, dx \, dy$$

where $\Delta \epsilon$ is the perturbation to the permittivity, approximately $\Delta \epsilon \approx 2n \delta n$ when $\delta n < n$. The longitudinal coefficient $K''_{jk}(z)$ is analogous to $K'_{jk}(z)$, but generally $K''_{jk}(z) \ll K'_{jk}(z)$ for the fiber modes, and thus this coefficient is usually neglected. Define two new coefficients

$$\sigma_{jk}(z) = \frac{\alpha n_{inc}}{2} \delta n_{inc}(z) \int \int \Delta \epsilon(x,y,z) \bar{e}_x(x,y) \cdot \bar{e}_y(x,y)$$

$$\kappa_{jk}(z) = \frac{\alpha n_{inc}}{4} \delta n_{inc}(z) \int \int \Delta \epsilon(x,y,z) \bar{e}_x(x,y) \cdot \bar{e}_y(x,y)$$

where $\sigma$ is the "dc" (period-averaged) coupling coefficient and $\kappa$ is the "ac" coupling coefficient, then the general coupling coefficient can be written

$$K'_{jk}(z) = \sigma_{jk}(z) + 2\kappa_{jk}(z) \cos \left[ \frac{2\pi}{\Lambda} z + \phi(z) \right]$$

Equations (2.4) to (2.7) are the coupled-mode equations that describe fiber grating spectra.

Near the wavelength for which reflection of a mode of amplitude $A(z)$ into an identical counter-propagating mode of amplitude $B(z)$ is the dominant interaction in a Bragg grating. Equations (2.4) can be simplified by retaining only terms that involve the amplitudes of the particular mode, and then making the "synchronous approximation"[4]. The latter amounts to neglecting terms on the right-hand sides of the differential equations that contain a rapidly oscillating $z$ dependence, since these contribute little to the growth and decay of the amplitudes. The resulting equations can be written

$$\frac{du}{dz} = j\tilde{\sigma}u(z) + j\kappa v(z)$$

$$\frac{dv}{dz} = -j\tilde{\sigma}v(z) - j\kappa u(z)$$

where the amplitudes $u$ and $v$ are $u(z) = A(z) \exp(i\tilde{\epsilon}z - \phi/2)$ and $v(z) = B(z) \exp(-i\tilde{\epsilon}z + \phi/2)$, $\tilde{\sigma}$ is a general "dc" self-coupling coefficient defined as

$$\tilde{\sigma} = \sigma + \kappa \frac{1}{2} \frac{d\phi}{dz}$$

where $\sigma$ is the "dc" (period-averaged) coupling coefficient. For a single-mode Bragg grating

$$\sigma = \frac{2\pi}{\Lambda} g \left[ \delta n_{inc}(z) + C \delta n_{inc}(z) \right]$$

$$\kappa = \kappa^* = \frac{\pi}{\Lambda} g \tilde{\delta n}_{inc}(z)$$
where \( g \) is the overlap integral of the guided mode in the photosensitivity region.

Note that for a real grating the index modulation function \( \delta n_{\text{eff}}(z) \) is not necessarily sinusoidal even if the UV pattern for grating writing is sinusoidal. This results from the nonlinear response of the fiber index change to the UV intensity and the saturation of index change. To analyze a grating with non-sinusoidal \( \delta n_{\text{eff}}(z) \), we need to find the Fourier transform of \( \delta n_{\text{eff}}(z) \) and extract the sinusoidal harmonic that corresponds to the wavelength that we are interested in. Other harmonics will make little net contribution to this wavelength [1]. Therefore, we can always assume the index modulation to be sinusoidal.

3. Solving the Couple Mode Equations

3.1. Uniform gratings

If the grating is uniform along \( z \), then the coupled mode equations are coupled first-order ordinary differential equations with constant coefficients, for which closed-form solutions can be found when appropriate boundary conditions are specified. The reflectivity of a uniform fiber grating of length \( L \) can be found by assuming a forward-going wave incident from \( z = -\infty \), passing through the grating and requiring that no backward-going wave exists for \( z \geq L \), i.e., \( u(L) = 1 \) and \( v(L) = 0 \). The amplitude and power reflection coefficients \( \rho = v(0)/u(0) \) and \( r = |\rho|^2 \), respectively, can then be shown to be

\[
\rho = \frac{-\kappa \sinh(\sqrt{\kappa^2 - \sigma^2} L)}{\sigma \sinh(\sqrt{\kappa^2 - \sigma^2} L) + i \sqrt{\kappa^2 - \sigma^2} \cosh(\sqrt{\kappa^2 - \sigma^2} L)}
\]

(3.1)

and

\[
r = \frac{\sinh^2(\sqrt{\kappa^2 - \sigma^2} L) - \sigma^2}{\cosh^2(\sqrt{\kappa^2 - \sigma^2} L) - \sigma^2/k^2}
\]

(3.2)

3.2. Non-uniform gratings

For non-uniform gratings, the couple mode equations must be solved by numerical methods. There are two approaches for calculating the grating spectra. One is direct numerical integration of the coupled mode equations. For a grating of length \( L \), one generally takes \( u(L) = 1 \) and \( v(L) = 0 \), and then integrates backward from \( z = L \) to \( z = 0 \), thus obtaining \( u(0) \) and \( v(0) \). Typically, adaptive-step-size Runge-Kutta numerical integration works well. The other approach is a piecewise-uniform approach, in which the grating is divided into a number of uniform pieces. The closed form solution for each uniform piece is combined by multiplying matrices associated with the pieces. This method is simple to implement, almost always sufficiently accurate, and generally the fastest [5].

The grating can be divided into \( M \) uniform sections. Define \( u_i \) and \( v_i \) to be the field amplitudes after traversing the section \( i \). Thus for Bragg grating, we start with \( u_0 = u(L) = 1 \) and \( v_0 = v(L) = 0 \) and calculate \( u(0) = u_M \) and \( v(0) = v_M \). The propagation through each uniform section \( i \) is described by a matrix \( F_i \), defined such that

\[
\begin{bmatrix}
  u_i \\
  v_i
\end{bmatrix}
= F_i
\begin{bmatrix}
  u_{i-1} \\
  v_{i-1}
\end{bmatrix}
\]

(3.3)

and

\[
F_i = \begin{bmatrix}
  \cosh(\gamma_B \Delta z) - j \frac{\tilde{\sigma}}{\gamma_B} \sinh(\gamma_B \Delta z) & -j \frac{\kappa}{\gamma_B} \sinh(\gamma_B \Delta z) \\
  j \frac{\kappa}{\gamma_B} \sinh(\gamma_B \Delta z) & \cosh(\gamma_B \Delta z) + j \frac{\tilde{\sigma}}{\gamma_B} \sinh(\gamma_B \Delta z)
\end{bmatrix}
\]

(3.4)

where \( \Delta z \) is the length of the \( i \)th uniform section, the coupling coefficients \( \tilde{\sigma} \) and \( \kappa \) are the local values in the \( i \)th section and
Once all of the matrices for the individual sections are known, the output amplitudes can be found from

\[
\begin{bmatrix}
    u_{M} \\
    v_{M}
\end{bmatrix}
= 
F
\begin{bmatrix}
    u_{0} \\
    v_{0}
\end{bmatrix}
\]

\[
F = F_{M} \cdot F_{M-1} \cdots F_{i} \cdots F_{1}
\]

The number of sections needed for the piecewise-uniform calculation is determined by the required accuracy. \( M \) may not be arbitrarily large, since the coupled-mode-theory approximations that lead to equations (2.8) are not valid when a uniform grating section is only a few grating periods long [5]. Thus we require

\[
L >> \Delta z >> \Lambda
\]

To implement the piecewise-uniform method for apodized and chirped gratings, we simply assign constant values \( \Lambda \), \( \sigma \) and \( \kappa \) to each uniform section, where these might be the \( z \)-dependent values of \( A(z) \), \( \alpha(z) \) and \( \kappa(z) \) evaluated at the center of each section.

For phase-shifted and sampled gratings, we insert a phase-shift matrix \( F_{\phi_{i}} \) between the factors \( F_{i} \) and \( F_{i+1} \) in the product in the above equation for a phase shift after the \( i \)th section. The phase-shift matrix is of the form

\[
F_{\phi_{i}} = \begin{bmatrix}
    \exp(-j\phi_{i}/2) & 0 \\
    0 & \exp(j\phi_{i}/2)
\end{bmatrix}
\]

Here, \( \phi_{i} \) is the shift in the phase of the grating.

According to coupled mode theory and the piecewise-uniform approach, we’ve developed a program that enables us to calculate the reflectivity, transmission, time group delay and dispersion of various types of fiber Bragg grating. This program can take into account of grating length, effective index of the fiber, arbitrary apodization profiles and arbitrary chirp profiles and can model fiber gratings fabricated under variable practical conditions. For sampled gratings

\[
\phi_{i} = \frac{2\pi n_{\text{eff}}}{\lambda} \Delta z_{o}
\]

where \( \Delta z_{o} \) is the separation between two grating sections.

Once we know the parameters such as \( L \), \( n_{\text{eff}} \), \( \delta n_{ac}(z) \), \( \delta n_{dc}(z) \), \( A(z) \), \( \phi(z) \) and phase shift of a grating, we can calculate the reflectivity, time group delay and dispersion of the grating.

### 3.3. Bragg grating’s Time group delay and dispersion

Recently, there is growing interest in the dispersion properties of fiber Bragg gratings for application such as dispersion compensation, pulse shaping, and fiber and semiconductor laser components. The time group delay and dispersion of the reflected light can be determined from the phase of the amplitude from the phase of the amplitude reflection coefficient \( \rho \). If we denote \( \theta_{\rho} = \text{phase}(\rho) \), then the delay time \( \tau_{\rho} \) (usually given in units of picoseconds) for light reflected off of a grating is

\[
\tau_{\rho} = \frac{d\theta_{\rho}}{d\omega} = -\frac{2\pi}{\lambda} \frac{d\theta_{\rho}}{d\lambda}
\]

The dispersion \( d_{\rho} \) (usually in ps/nm) is the rate of change of delay time with wavelength

\[
d_{\rho} = \frac{d\tau_{\rho}}{d\lambda} = \frac{2\tau_{\rho}}{\lambda} \frac{d^{2}\theta_{\rho}}{d\lambda^{2}} = \frac{2\pi}{\lambda} \frac{d^{2}\theta_{\rho}}{d\omega^{2}}
\]
4. DWDM Gratings for 100 GHz Application

4.1. Design object

The DWDM grating for 100 GHz application is a non-chirped, apodized grating with the specifications shown as follows:

- Center wavelength: ~ 1550 nm.
- Bandwidth at –0.5 dB: 0.4 nm
- Bandwidth at –30 dB: 1.1 nm
- Transmission at –25 dB: 0.2 nm
- Channel isolation: > 30 dB
- Reflectivity: > 99.9%

The grating length should be short so as to be easy to be packaged.

4.2. Grating length and index change

For a uniform Bragg grating, from equation (3.2), we see that the maximum reflectivity \( r_{max} \) is:

\[
r_{max} = \tanh^2(\kappa L)
\]

and it occurs when \( \sigma = 0 \) at the wavelength

\[
\lambda_{max} = \left[ 1 + \frac{g(\partial n_\omega + C \partial n_\omega)}{n_{eff}} \right] \lambda_D
\]

From equation (4.1), the maximum reflectivity of a Bragg grating is determined by the index change together with the grating length. Define \( \Delta \lambda_0 \) as the bandwidth that between the first zero on either side of the maximum reflectivity. From equation (3.2), we find

\[
\frac{\Delta \lambda_0}{\lambda} = \frac{g \partial n_\omega}{n_{eff}} \sqrt{1 + \left( \frac{\lambda_D}{g \partial n_\omega L} \right)^2} \tag{4.3}
\]

in the “weak-grating limit”, for which \( g \partial n_\omega \) is very small, we find

\[
\frac{\Delta \lambda_0}{\lambda} \rightarrow \frac{\lambda_D}{n_{eff} L} \left( g \partial n_\omega \ll \frac{\lambda_D}{L} \right) \tag{4.4}
\]

the bandwidth of weak grating is mainly determined by the grating length. However, in the “strong grating limit”, we find

\[
\frac{\Delta \lambda_0}{\lambda} \rightarrow \frac{g \partial n_\omega}{n_{eff} L} \left( g \partial n_\omega \gg \frac{\lambda_D}{L} \right) \tag{4.5}
\]

In strong grating, the light does not penetrate the full length of the grating, and thus the bandwidth is independent of length and directly proportional to the induced index change.

The DWDM grating is a kind of strong grating. Therefore, when the grating length is fixed, the reflectivity bandwidth at –0.5dB is mainly determined by index change. Figure 2 shows the reflectivity spectrum of uniform gratings with different index change ranged from 1x10^-5 to 5x10^-4 while grating length is 14 mm.
4.3. Apodization profile

A uniform fiber grating has two ends. Thus, it begins abruptly and ends abruptly. The Fourier transform of such a “rectangular” function immediately yields the well-known sinc function, with its associated side-lobes structure apparent in the reflection spectrum. The transform of a Gaussian function, for example, is also Gaussian, with no side lobes. A grating with a similar refractive modulation amplitude profile diminishes the side lobes substantially. The suppression of the side lobes in the reflection spectrum by gradually increasing the coupling coefficient with penetration into, as well as gradually decreasing on exiting from, the grating is called apodization. The benefits of apodization are not manifest only in the smoothness of the reflection spectrum, but also in the dispersion characteristics. Apodization can essentially reduce the time group delay level of a chirp grating.

However, simply changing the refractive index modulation amplitude changes local Bragg wavelength as well, forming a distributed Fabry-Perot interferometer, which causes structure to appear on the blue side of the reflection spectrum of the grating, although side-lobe amplitudes are reduced [5]. To avoid this complication, the key is to maintain an unchanging average refractive index throughout the length of the grating while gradually altering the refractive index modulation amplitude. Figure 3 compares the gratings that (a) without apodized, (b) apodized with non-zero DC index change and (c) apodized with zero DC index change.

To obtain a zero DC index change grating, usually double scan process are needed in the grating writing. In the first scan, the UV beam passes through the phase mask and writes the grating into the fiber. The intensity of the UV beam varies when scans along the fiber to generate an apodization profile. Then, scan the UV beam along the fiber again, illuminating the fiber directly by removing the phase mask away. This time the intensity of the UV beam varies in contrary to first scan so that every point of the grating receives the same dose of total UV exposes in both scans.
There are several most commonly used functions can be used as apodization profiles. Such as Gauss/super-Gauss profile

\[
\hat{n}(z) = \hat{n}_{\text{max}} \cdot e^{-\left[\frac{(z-0.5L)}{0.5L} \cdot \ln\left(\frac{\hat{n}_{\text{max}}}{\hat{n}_{\text{max}}}\right)\right]^2}
\]

where \(\hat{n}_{\text{max}}\) is the maximum index change and this function is truncated at \(\hat{n}(z) = \hat{n}_{\min}\).

“Raised-cosine” profile

\[
\hat{n}(z) = 0.5 \cdot \hat{n}_{\text{max}} \cdot \left[1 + \cos\left(\pi \frac{z - 0.5L}{0.5L}\right)\right]
\]

Sinc function profile

\[
\hat{n}(z) = \hat{n}_{\text{max}} \cdot \left[\sin\left(0.5\pi \left(\frac{z - 0.5L}{0.5L}\right)^{\alpha}\right)\right]^{\beta}
\]

Tanh profile

\[
\hat{n}(z) = \hat{n}_{\text{max}} \cdot \left[1 + \tanh\left(\frac{2\pi(z - 0.5L)}{0.5L}\right)\right]^{\alpha}\quad (\alpha > 0)
\]

Blackman profile

\[
\hat{n}(z) = \hat{n}_{\text{max}} \cdot \left[1 + (1 + B) \cos\left(\frac{\pi(z - 0.5L)}{0.5L}\right) + B \cos\left(\frac{2\pi(z - 0.5L)}{0.5L}\right)\right] \div 2 + 2B
\]

Cauchy profile

Figure 4 shows a comparison of the reflectivity spectrum of various types of apodization profile. Note that both Gaussian function and sinc function must be truncated at certain index change level at the ends of the grating while Raised-cosine profile can be truncated at zero index change at the ends, therefore the raised-cosine profile can result in a better isolation. It shows that except the truncation effect, the type of apodization profile doesn't change the reflectivity spectrum a lot. As we will show in the following sections, practically, the real apodization profile is impossible as smooth as a math function we use in Figure 4, and therefore the spectrum isolation mainly depends on the smoothness of the real index change profile.

Figure 4. Reflectivity spectrum of gratings with various types of apodization profiles.

\(L = 14\text{mm}, \quad \hat{n}_{\text{max}} = 5 \times 10^{14}\)
5. Chirped Fiber Bragg Gratings

5.1. Introduction

Gratings that have a nonuniform period along their length are known as chirped gratings. The chirp may be linear, may be quadratic, or may even have jumps in the period. A grating could also have a period that varies in any function along its length. Chirped gratings have many applications. In particular, the linearly chirped grating has been found that it can be used as a dispersion compensation device. There are several parameters that affect the performance of chirped fiber Bragg gratings for dispersion compensation. These are the dispersion, bandwidth and deviations from linearity of the group delay and group delay ripple.

5.2. Grating length and bandwidth

We consider a chirped grating with linear delay characteristics, over a bandwidth of $\Delta \lambda_{chirp}$. Generally, the parameters of importance for chirped dispersion compensation gratings are the target center wavelength $\lambda_0$, bandwidth, reflectivity, dispersion $D$, group delay $G$ and the linearisation of the delay ripple. According to the used fiber parameter, such as $n_{eff}$ of the fiber, parameters that need to be calculated for the design. such as grating center period $\Lambda_0$, period bandwidth of the grating, i.e. $\Lambda_j - \Lambda_z$, and total length of the grating $L$, could be determined.

According to the Bragg condition, the center period of grating is easy to be determined.

$$\lambda_0 = 2\Lambda_0 n_{eff}$$  \hspace{1cm} (5.1)

Then, the bandwidth of the grating $\Lambda_j - \Lambda_z$ can also be easily determined. For non-apodized gratings, the group time delay can be roughly evaluated by

$$G = 2n_{eff} L/c$$  \hspace{1cm} (5.2)

Where $c$ is light speed in vacuum. And then the dispersion can be evaluated by

$$D = G/\Delta \lambda_{chirp} = \frac{2n_{eff} L}{c\Delta \lambda_{chirp}}$$  \hspace{1cm} (5.3)

Practically, in order to get lower ripples in the group time delay curve, the grating must be apodized. The apodization requires an extra length at both grating ends, so the total grating length has to be longer than the length evaluated in equation (5.3). As a matter of fact, the grating length $L$ determined in equation (5.2) and (5.3) can be treated as the effective length of the grating.

5.3. Apodization profile

Similar to the WDM gratings, the immediate effect of apodization on chirped gratings is the dramatic reduction in the side-lobe levels in the reflection spectrum. Another advantage of apodization is in the reduction of internal interference effects that cause the group delay to acquire a ripple.

Figure 9 shows the reflectivity (left) and group delay (right) of 107mm long raised cosine flat top apodized (bottom) and non-apodized Gratings (up). The apodization not only can dramatically improve the reflectivity spectrum isolation, but also essentially reduce the group delay ripple. With non-apodized, the group delay ripple is about ±50ps with high frequency, with apodized, the ripple is reduced to a few picoseconds with very low frequency.
Figure 9. Reflectivity (left) group delay (right) of 107mm long of raised cosine flat top apodized (bottom) and non-apodized Gratings (up).

Similar to WDM grating, practically, the real index change profile of the chirped gratings can not be as smooth as the ideal mathematical apodization function. Again, Take the luminescence data as a “real” $dn$ profile and calculate it again, we find, similar the WDM gratings, the reflectivity spectrum isolation reduced to about $-33\text{dB}$ and group delay ripple will raise. Figure 10 shows the simulation reflectivity, group delay and group delay ripple by use of the luminescence profile.

Figure 10. The simulation result of reflectivity (a), group delay (c) and group delay ripple (d) by use of the luminescence profile (b).

If there is strong pulse in the index change profile caused by some unknown reasons, this pulse will greatly reduce the reflectivity spectrum isolation and raise the ripple. Figure 11 is such an example. Comparing with the result in Figure 10, the ripple is raised from about $\pm 20\text{ps}$ to $\pm 80\text{ps}$. 
6. Tilted Fiber Gratings

Radiation-mode coupling can be used to advantage for fiber taps, spectrum flattening, filtering of amplified spontaneous emission, and pump-light rejection in optically amplified optical communications systems. A series of papers of Mizrahi, Sipe and Erdogan have described the basic formalism for understanding bound-mode to radiation-mode coupling in a tilted fiber grating [6,9,10].

Radiation-mode coupling can be enhanced and to some extent controlled if a tilt is provided in the fringes of the grating. With simple symmetry arguments, a LP01 bound mode in an untilted grating can couple only to LP radiation modes with azimuthal quantum numbers 0 and 2; in the presence of a tilted grating coupling to all odd radiation modes and all other even radiation modes is allowed as well. In addition simply to enhancing the maximum radiation-mode coupling, variation of grating tilt affects the width of the loss spectrum, the separation of the wavelength region at which maximum radiation-mode coupling occurs from that at which Bragg reflection occurs, and the Bragg reflection spectrum.

In a normal step-index fiber with a finite glass cladding, coupling can occur between the bound core mode and the bound cladding modes of the fiber. These effects are not considered here; we consider only coupling between the bound core mode and the continuum of radiation modes in a fiber with an infinite cladding. Experimentally, the behavior of a bound core mode in an infinite-cladding fiber can be approximately realized by a fiber immersed in or coated with a medium of refractive index equal to or higher than that of the cladding.

6.1. Mode amplitude equations

Suppose \( \Delta n(x,y,z) \) is a spatially varying effective refractive index and \( \bar{n} \) is a reference refractive index, and write

\[
\frac{\Delta n(x,y,z)}{\bar{n}} = \zeta(x,y)\eta(z')
\]

(6.1)

where \( \eta(z') \) is a function that specifies the index variation in the fiber core, \( \zeta(x,y) \) is equal to unity in the fiber core and vanishes elsewhere. We consider a geometry in which the wave vector associated with the
interference of the two beams makes an angle $\theta$ from the fiber axis and lies in a plane that we take to be the x-z plane (see Figure 13). Thus the natural variable for the function $\eta(z')$ is $z'=z\cos\theta+x\sin\theta$. We also wish to allow for possible chirping of the grating. Thus, for the function $\eta(z')$, we take

$$\eta(z') = \sigma(z') + 2\kappa(z') \cos[2K_g z' + \phi(z')]$$

(6.2)

Here $2K_g$ is the nominal wave number of the grating, and the function $\phi(z')$, assumed to be slowly varying on a length scale set by the nominal wave number, can describe a position-dependent chirp. The other slowly varying functions, $2\kappa(z')$ and $\sigma(z')$, describe the grating amplitude and the concomitant slowly varying perturbation in the background index of refraction that accompanies the grating writing.

Define corresponding unbarred functions,

$$\sigma(z) = \sigma(z \cos \theta) \quad \phi(z) = \phi(z \cos \theta)$$

$$\kappa(z) = \kappa(z \cos \theta) \quad \kappa(z) = \kappa(z) \exp[\phi(z)]$$

(6.3)

The coupled-amplitude equations can be written as follow. The type of modes here can be labeled by $(\alpha, \rho)$. The variable $\alpha$ can be discrete (indicated by $m$) or continuous (indicated by $\rho$) and characterizes the propagation of the mode in the $z$ direction; $\rho$ specifies any other degeneracy indices. Thus, to specify a mode and a direction (propagation in the $+z$ or $-z$ direction), we need a sing ($\pm$) as well as $(\alpha, \rho)$. For modes going to the right ($+z$),

$$\begin{align*}
\overline{b}^{-1} \frac{da^{+}_{\alpha}}{dz} &= i\sigma(z) \left[ \sum_{\alpha'} g^{+\alpha^*}_{\alpha \alpha'} a^{+\alpha'}_{\alpha'}(z) e^{i(\beta_{\alpha'} \rho_{\alpha'} z)} + \sum_{\alpha'} g^{-\alpha^*}_{\alpha \alpha'} a^{-\alpha'}_{\alpha'}(z) e^{i(-\beta_{\alpha'} \rho_{\alpha'} z)} \right] \\
&+ i\kappa_{\alpha}(z) e^{iK \rho_{\alpha} z} \left[ \sum_{\alpha'} \mu^{+\alpha^*}_{\alpha \alpha'} a^{+\alpha'}_{\alpha'}(z) e^{i(\beta_{\alpha'} \rho_{\alpha'} z)} + \sum_{\alpha'} \mu^{-\alpha^*}_{\alpha \alpha'} a^{-\alpha'}_{\alpha'}(z) e^{i(-\beta_{\alpha'} \rho_{\alpha'} z)} \right] \\
&+ i\kappa_{\alpha}(z) e^{-i2K \rho_{\alpha} z} \left[ \sum_{\alpha'} \nu^{+\alpha^*}_{\alpha \alpha'} a^{+\alpha'}_{\alpha'}(z) e^{i(\beta_{\alpha'} \rho_{\alpha'} z)} + \sum_{\alpha'} \nu^{-\alpha^*}_{\alpha \alpha'} a^{-\alpha'}_{\alpha'}(z) e^{i(-\beta_{\alpha'} \rho_{\alpha'} z)} \right]
\end{align*}$$

(6.4)

and for the modes going to the left ($-z$),

$$\begin{align*}
\overline{b}^{-1} \frac{da^{-}_{\alpha}}{dz} &= -i\sigma(z) \left[ \sum_{\alpha'} g^{+\alpha^*}_{\alpha \alpha'} a^{+\alpha'}_{\alpha'}(z) e^{i(\beta_{\alpha'} \rho_{\alpha'} z)} + \sum_{\alpha'} g^{-\alpha^*}_{\alpha \alpha'} a^{-\alpha'}_{\alpha'}(z) e^{i(-\beta_{\alpha'} \rho_{\alpha'} z)} \right] \\
&- i\kappa_{\alpha}(z) e^{iK \rho_{\alpha} z} \left[ \sum_{\alpha'} \mu^{+\alpha^*}_{\alpha \alpha'} a^{+\alpha'}_{\alpha'}(z) e^{i(\beta_{\alpha'} \rho_{\alpha'} z)} + \sum_{\alpha'} \mu^{-\alpha^*}_{\alpha \alpha'} a^{-\alpha'}_{\alpha'}(z) e^{i(-\beta_{\alpha'} \rho_{\alpha'} z)} \right] \\
&- i\kappa_{\alpha}(z) e^{-i2K \rho_{\alpha} z} \left[ \sum_{\alpha'} \nu^{+\alpha^*}_{\alpha \alpha'} a^{+\alpha'}_{\alpha'}(z) e^{i(\beta_{\alpha'} \rho_{\alpha'} z)} + \sum_{\alpha'} \nu^{-\alpha^*}_{\alpha \alpha'} a^{-\alpha'}_{\alpha'}(z) e^{i(-\beta_{\alpha'} \rho_{\alpha'} z)} \right]
\end{align*}$$

(6.5)

Here, $\overline{b} = \frac{\alpha}{c} \overline{n}$. The coupling constants appearing in Eq.(6.4) and (6.5) that mediate the interaction between the modes that is due to the slowly varying perturbation in the background index of refraction are

$$g^{+}_{\alpha \alpha'} = \frac{2\alpha \epsilon \epsilon_0}{\sqrt{P_\alpha P_{\alpha'}}} \int E^{+\text{f}}_\alpha(x, y) \cdot \chi(x, y) E^{-\text{f}}_{\alpha'}(x, y) dx$$

(6.6)

where $i$ and $j$ can be $+$ or $-$. Likewise, the coupling constants that mediate the interaction that is due to the $+2K$ component of the grating are given by
and those that mediate the interaction that is due to the $-2K$ component of the grating are given by
\[ v'_{\omega} = \frac{2\pi c e_0}{\sqrt{P_a P_o}} \int E_{\omega}^*(x,y) \exp[-2iKx \tan \theta] \cdot \zeta(x,y) E_{\omega} (x,y) \, ds \]  
\tag{6.8}

For an untilted grating ($\theta = 0$) we have $g_{\omega} = \mu_{\omega} = v'_{\omega}$, but more generally the constants can be quite different.

6.2. Bragg scattering

The simplest scattering geometry involves a grating wave number $2K$ adjusted to scatter a discrete mode propagation to the right ($+z$) to the corresponding discrete mode propagating to the left ($-z$). Of primary interest is the lowest (and often only) discrete mode $LP_{01}$. Here we use the discrete index $\alpha = m = (01)$, reserving the index $p$ to label the two polarizations. In the simplest case of untilted gratings, ($\theta = 0$), symmetry prohibits scattering from a mode of one polarization to that of another. Then the coupling constants that will be of interest are
\[ g_{01,01} = g_{01,01}^* = \frac{2\pi c e_0}{P_o} \int E_{01}^*(x,y) \zeta(x,y) E_{01} (x,y) \, ds = g_f, \]  
\[ \mu_{01,01} = (\mu_{01,01}^*)^* = \frac{2\pi c e_0}{P_o} \int E_{01}^*(x,y) \zeta(x,y) E_{01} (x,y) \, ds = g_b, \]  
\tag{6.9}

where both $g_f$ and $g_b$ are real. In the usual approximation of keeping only the phase-matched terms, equations (6.4) and (6.5) reduce to
\[ \bar{\beta} \cdot \frac{da}{dz} = ig_f \sigma(z) a(z) + ig_s \kappa(z) \exp[i\phi(z)] \exp[2i(K - \beta_0)z] \]  
\[ \bar{\beta} \cdot \frac{dv}{dz} = -ig_b \sigma(z) v(z) - ig_s \kappa(z) \exp[-i\phi(z)] \exp[-2i(K - \beta_0)z] \]  
\tag{6.10}

Introducing new variables,
\[ u(z) = a(z) \exp[i\phi(z)] \exp[-i(K - \beta_0)z] \]  
\[ v(z) = a(z) \exp[i\phi(z)] \exp[(K - \beta_0)z] \]  
\tag{6.11}

Equations (6.10) take the form
\[ \bar{\beta} \cdot \frac{du}{dz} = i[g_f \sigma(z) + \delta - \frac{1}{2} \bar{\beta}^{-1} \frac{d\phi}{dz}] u(z) + ig_s \kappa(z)v(z) \]  
\[ \bar{\beta} \cdot \frac{dv}{dz} = -i[g_f \sigma(z) + \delta - \frac{1}{2} \bar{\beta}^{-1} \frac{d\phi}{dz}] v(z) - ig_s \kappa(z)u(z) \]  
\tag{6.12}

where $\delta = (\beta_0 - K) / \bar{\beta} = (\omega - \omega_{\text{Bragg}}) / \omega_{\text{Bragg}}$ specifies the detuning from the Bragg resonance condition. In the weakly guiding approximation we have $\beta_0^* = k_n^2 n_s^2 + b_0^* (k_0^2 - k_n^2)$, where for $LP_{01}$ modes, $b_0^*$ is a solution to the dispersion relation
\[ V \sqrt{1 - b} \frac{J_{1,1}(V \sqrt{1 - b})}{J_{0,1}(V \sqrt{1 - b})} = -V \sqrt{b} \frac{K_{1,1}(V \sqrt{1 - b})}{K_{0,1}(V \sqrt{1 - b})} \]  
\tag{6.13}

where $J$ is a Bessel function of the first kind, $K$ is a modified Bessel function of the second kind, $V = (2\pi / \lambda) \sqrt{n_s^2 - n_0^2}$ is the $V$ number of the mode, with $a$ the core radius.

Equations (6.12) take the form of the usual coupled-mode equations, except for the presence of the factors $g_f$ and $g_b$. In the weakly guiding limit the expression (6.9) for $g_f$ and $g_b$ can be evaluated with use of the $LP_{01}$ mode profiles,
\[ g_r = g + \tilde{g} \quad g_s = g - \tilde{g} \]  
\[ g = b_{01}\left[ J_0^2(\kappa_{01}a) + 1 \right] \]  
\[ \tilde{g} = \Delta b_{01}\left(1 - b_{01}\right) \frac{1}{1 + 2\Delta b_{01}} \left[ 1 - \frac{J_1(\kappa_{01}a)J_1(\kappa_{01}a)}{J_1^2(\kappa_{01}a)} \right] \]  
where \( \Delta = (n_{\text{ev}} - n_{\text{eo}})/n_{\text{eo}} \), \( \kappa_{01}a = V\sqrt{1 - b_{01}} \).

Now we turn to the more complicated problem of tilted gratings. Referring back to equations (6.6) to (6.8), we see that \( g_r \) is unmodified by a nonzero \( \theta \), but \( g_b \) is not. The new expression of \( g_b \), which must be used in the coupled-mode equations (6.12), is

\[ \mu_{01,01}^- = \left( \mu_{01,01}^+ \right)^* = \frac{2\pi c e_0}{P_{01}} \int E_0^*(x,y) \exp\left[2iKx(\tan \theta)\right] \zeta(x,y)E_{01}(x,y) dx 
\]

For s-polarized mode,

\[ \frac{(\kappa_{01}a)^2 J_1^2(\kappa_{01}a)g_{s}}{2b_{01}} \]

\[ = \int_0^{\phi_{01}} J_0^2(\kappa_{01}a)J_0(\Omega u) du + \frac{2\Delta(1 - b_{01})}{1 + 2\Delta b_{01}} \int_0^{\phi_{01}} \frac{J_1(\Omega u)}{\Omega u} J_1^2(\kappa_{01}a) du \]

and for the p-polarized mode,

\[ \frac{(\kappa_{01}a)^2 J_1^2(\kappa_{01}a)g_{p}}{2b_{01}} \]

\[ = \int_0^{\phi_{01}} J_0^2(\kappa_{01}a)J_0(\Omega u) du + \frac{2\Delta(1 - b_{01})}{1 + 2\Delta b_{01}} \int_0^{\phi_{01}} \frac{J_1(\Omega u)}{\Omega u} J_1^2(\kappa_{01}a) du \]

where \( \Omega = (2K \tan \theta)/\kappa_{01} \).

Equations (6.12) satisfy energy conservation in the form

\[ \frac{d}{dz} \left[ \mu(z) \gamma(z) \right] = 0 \]  

6.3. Radiation-mode coupling

We begin by returning to the general equations (6.4) and (6.5) and collecting the relevant (i.e., phase matched) terms. We use the notation (6.11) to describe the discrete modes, assuming that both forward- and backward-propagating modes will be present as a result of possible Bragg scattering. Keeping the appropriate phase-matched terms, we have

\[ \frac{d}{dz} a_{\alpha} = -i\kappa(z) \exp\left[\frac{1}{2} i\phi(z)\right] \sum_{\mu,\nu} \mu_{\alpha,\mu,\nu} a_{\nu} \exp\left[i(\beta - \kappa)z\right] \]  

\[ \frac{d}{dz} a_{\alpha} = i\kappa(z) \exp\left[\frac{1}{2} i\phi(z)\right] \sum_{\mu,\nu} \mu_{\alpha,\mu,\nu} a_{\nu} \exp\left[-i(\beta - \kappa)z\right] \]  

for the radiation modes, where here we reserve \( \alpha \) as a label for the radiation modes (with propagation constant \( \beta \)), designating the discrete mode explicitly by \( (01) \) (with propagation constant \( \beta_{01} \)). Adding the radiation modes that can be phase matched in to equations (6.9), we replace equations (6.12) by

\[ \frac{d}{dz} \gamma(z) = g_r \sigma(z) + \delta - \frac{1}{2} \beta^{-1} \frac{d\phi}{dz} \]  

\[ \gamma(z) = g_r \sigma(z) + \delta - \frac{1}{2} \beta^{-1} \frac{d\phi}{dz} \]
Consider first equation (6.19). The philosophy of the calculation is that, like the Bragg scattering, the radiation-mode coupling will affect the amplitudes \( u(z) \) and \( v(z) \) only over distances that are much greater than \( \bar{\beta}^{-1} \). Thus, with respect to solving equations (6.19), over distances of the order of a few \( \bar{\beta}^{-1} \) from \( z = z_0 \), we may write

\[
\begin{align*}
  u(z) &\approx u(z_0) \exp\left[\frac{1}{2}i\phi(z)\right] \exp\left[\left(\beta - \bar{\beta}\right)z\right] v^+_{a;01}(z_0) u(z) \\
v(z) &\approx v(z_0) \exp\left[-\frac{1}{2}i\phi(z)\right] \exp\left[-\left(\beta - \bar{\beta}\right)z\right] v^+_{a;01}(z_0) v(z)
\end{align*}
\]

(6.22)

where \( \gamma(z) = \gamma(z_0) \). Using relations (6.22) in equations (6.19), we can find the particular solution for \( a^+_a \).

Then, using relation (6.22) again, we can regroup the terms to find, for example for \( a^+_a \),

\[
\begin{align*}
  a^+_a(z) &= -\frac{\bar{\beta}k(z)\exp\left[-\frac{1}{2} \phi(z)\right] \exp\left[i(\beta - K)z\right] v^+_{a;01} u(z)}{\beta - \bar{\beta}_{\text{res}}(z)} \\
  \text{where }
  \beta_{\text{res}}(z) &= K - \bar{\gamma}^2(z) + \frac{1}{2} \frac{d\phi}{dz} = \left(2K + \frac{d\phi}{dz}\right) - \left[\beta_{\text{01}} + \bar{\gamma}^2(z)\right]
\end{align*}
\]

(6.23)

(6.24)

Using equation (6.23) and the corresponding equation for \( a^+_a \) in equation (6.20), we have

\[
\begin{align*}
  \bar{\beta}^{-1} \frac{du}{dz} &= [\gamma(z) - \bar{\gamma}^2(z)\alpha(z) + igk(z) + \kappa(z)] u(z) \\
  \bar{\beta}^{-1} \frac{dv}{dz} &= -[\gamma(z) - \bar{\gamma}^2(z)\alpha(z) - igk(z) - \kappa(z)] v(z)
\end{align*}
\]

(6.25)

where

\[
A(z) = i\bar{\beta} \sum_{\rho_p} \frac{v^+_{p;01}(z)}{\beta - \bar{\beta}_{\text{res}}(z)} = \sum_{\rho} \left(\frac{\beta_{\rho}}{\rho_{\text{res}}} \bar{\beta}\right) v^+_{\rho;01}(z) \equiv \sum_{\rho} A_{\rho}(z)
\]

(6.26)

The sum over \( \rho \) is really an integral over \( \beta \). \( A(z) \) is positive and real. It is an effective extinction coefficient for the discrete mode and \( A_{\rho}(z) \) is the contribution to \( A(z) \) from the modes of degeneracy index \( \rho \). \( A(z) \) and \( A_{\rho}(z) \) depend on \( z \) only through the dependence of \( \beta_{\text{res}} \) on \( z \). A bound mode that are either s- or p-polarized with respect to the grating. For a radiation mode of type LP\( q \), \( q = 0, 1, 2, \ldots \), then there are four coefficients to consider: \( A_{ij}^{q,-} \), as \( i \) and \( j \) vary over \( s \) and \( p \). Using in equation (6.26) the expressions for the bound and radiation modes in the LP approximation, we can put the extinction coefficients in the form

\[
A_{ij}^{q,-}(z) = 4\left(\frac{\beta_{\rho}}{\rho_{\text{res}}} \bar{\beta}\right)^q \frac{J_1(\kappa_{\rho} a)}{J_1(\kappa_{\rho} a)} \left| c_{ij}^{q,-}\right|^2
\]

(6.27)

where \( e_\rho = 2 \) if \( q = 0 \), and \( e_\rho = 1 \) otherwise, and

\[
\Gamma_q = \left[\alpha(\rho a) J_{\rho}(\rho a) \left(\rho a\right)^{\rho+1} J_{\rho+1}(\rho a) \right]^2 + \left[\alpha(\rho a) J_{\rho}(\rho a) N_{\rho}(\rho a) \right] \left(\rho a\right)^{\rho+1} J_{\rho+1}(\rho a) N_{\rho+1}(\rho a)\right]^{-1}
\]

(6.28)

with

\[
\rho = \sqrt{n_{\rho}^2 k^2 - \beta^2}
\]

(6.29)

and where \( N_{\rho} \) is the Bessel function of the second kind of order \( q \). The terms \( c_{ij}^{q,-} \) are integrals of products of Bessel functions over the core. They are all of the form

\[
c_{ij}^{q,-} = \frac{1}{2(\kappa_{\rho} a)^2} \int_{\kappa_{\rho} a}^1 \left| f_{ij}^{q,-}(u)\right| du
\]

(6.30)
\[
I_q^{s-c}(u) = 2 \left( \frac{n_{\omega} k}{\beta} + 1 \right) J_q(Tu) J_0(u) J_q(\Omega u) \\
+ \left( \frac{n_{\omega} k}{\beta} - 1 \right) [J_{q+2}(Tu) J_0(u) J_{q+2}(\Omega u) + J_{q-2}(Tu) J_0(u) J_{q-2}(\Omega u)] \\
- \frac{\kappa_{\omega_1} \rho}{\beta_{\omega_1}} \left\{ J_{q+1}(Tu) J_1(u) [J_q(\Omega u) + J_{q+2}(\Omega u)] - J_{q-1}(Tu) J_1(u) [J_q(\Omega u) + J_{q-2}(\Omega u)] \right\} 
\]

(6.31)

is the integral involved in calculating the coupling of a LP_{01} mode that is s-polarized with respect to the grating to an s-polarized LP_q radiation mode. Similarly

\[
I_q^{p-c}(u) = 2 \left( \frac{n_{\omega} k}{\beta} + 1 \right) J_q(Tu) J_0(u) J_q(\Omega u) \\
- \left( \frac{n_{\omega} k}{\beta} - 1 \right) [J_{q+2}(Tu) J_0(u) J_{q+2}(\Omega u) + J_{q-2}(Tu) J_0(u) J_{q-2}(\Omega u)] \\
- \frac{\kappa_{\omega_1} \rho}{\beta_{\omega_1}} \left\{ J_{q+1}(Tu) J_1(u) [J_q(\Omega u) - J_{q+2}(\Omega u)] - J_{q-1}(Tu) J_1(u) [J_q(\Omega u) - J_{q-2}(\Omega u)] \right\} 
\]

(6.32)

\[
I_q^{p-c}(u) = 2 \left( \frac{n_{\omega} k}{\beta} + 1 \right) J_q(Tu) J_0(u) J_q(\Omega u) \\
+ \left( \frac{n_{\omega} k}{\beta} - 1 \right) [J_{q+2}(Tu) J_0(u) J_{q+2}(\Omega u) - J_{q-2}(Tu) J_0(u) J_{q-2}(\Omega u)] \\
+ \frac{\kappa_{\omega_1} \rho}{\beta_{\omega_1}} \left\{ J_{q+1}(Tu) J_1(u) [J_q(\Omega u) - J_{q+2}(\Omega u)] + J_{q-1}(Tu) J_1(u) [J_q(\Omega u) - J_{q-2}(\Omega u)] \right\} 
\]

(6.33)

(6.34)

In these expressions, \( \Omega \) and \( T \) are given by

\[
\Omega = \frac{2K \tan \theta}{\kappa_{\omega_1}}, \quad T = \frac{\tau}{\kappa_{\omega_1}} 
\]

(6.35)

Field amplitudes \( u(z) \) and \( v(z) \) satisfying equations (6.25) no longer satisfy the energy conservation condition (6.18). Instead, they satisfy

\[
\frac{d}{dz} \left[ |u(z)|^2 - |v(z)|^2 \right] = -2\beta k^2(z) \Delta(z) \left[ |u(z)|^2 - |v(z)|^2 \right] 
\]

(6.36)

showing that the flux of light through the fiber decreases along the direction of energy propagation on account of scattering of light out into the radiation modes.

References


