TECHNICAL NOTE

VEHICLE ROUTING WITH TIME WINDOWS: TWO OPTIMIZATION ALGORITHMS

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We describe two optimization methods for vehicle routing problems with time windows. These are a K-Tree relaxation with time windows added as side constraints and a Lagrangian decomposition in which variable splitting is used to divide the problem into two subproblems—a semi-assignment problem and a series of shortest path problems with time windows and capacity constraints. We present optimal solutions to problems with up to 100 customers.

This paper is concerned with the vehicle routing problem with time window constraints (VRPTW) which can be characterized by the following parameters:

m = number of vehicles,

n = number of customers; index 0 denotes the depot,

Q_k = capacity of vehicle k,

q_i = demand of customer i,

c_{ij} = cost of travel from customer i to j,

t_{ij} = time for travel from customer i to j,

s_i = service time at customer i,

e_i = earliest time allowed for beginning delivery at customer i,

u_i = latest time allowed for beginning delivery at customer i,

$N = \{1, \ldots, n\}$,

$N_0 = N \cup \{0\}$, and

$M = \{1, \ldots, m\}$.

We are required to assign each customer to a vehicle and to sequence the set of customers assigned to each vehicle so as to minimize cost subject to vehicle capacity constraints and the requirement that the time for beginning delivery at customer i lies in the interval (e_i, u_i). We assume that if a vehicle arrives too early at a customer location, it will wait until the time window "opens."

Surveys on the literature on vehicle routing with time window constraints are given in Solomon and Desrosiers (1988) and in Desrochers et al. (1988).

The exact approaches we are aware of can be divided into:

(i) **Approaches based on dynamic programming.** This line of research has been followed by Kolen et al. (1987) and can be regarded as an extension of the Christofides et al. (1981) state-space relaxation method to problems with time windows. Problems with up to 15 customers have been solved to optimality.

(ii) **Approaches based on column generation.** In this class, Desrochers et al. (1992) presented an exact method with the capability of solving some 100-customer problems. The algorithm is based on a combination of linear programming relaxed set covering and column generation. This is currently the best approach.

(iii) **Lagrangian decomposition based methods.** Jörnsten et al. (1986), Madsen (1988, 1990), and Halse (1992) have applied various Lagrangian decomposition schemes to the vehicle routing problem with time windows in order to produce lower bounds. They are currently capable of solving some 100 customer problems to optimality using a combination of Lagrangian decomposition and branch-and-bound.

(iv) **K-tree based methods.** Fisher (1994) has extended the 1-tree method to a K-tree method for the classical vehicle routing problem and the VRPTW. No computational results are reported concerning the latter case.

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The first three approaches rely on the solution of a shortest path problem with time window (SPTW) and vehicle capacity constraints either to generate columns or as a part of a Lagrangian relaxation.

The results presented in this paper are based on further extensions of the work on Lagrangian decomposition reported above in (iii) and an extension of Fisher's (1994) exact K-tree algorithm for the classical vehicle routing problem to the case with time window constraints.

Until now only the method described in Desrochers et al. (1992) has been able to solve to optimality VRPTW of size up to 100 customers. The main contribution of our paper is to describe two new methods which are also able to solve to optimality problems of the same size.

1. SOLUTION METHODS

1.1. The Lagrangian Relaxation/Variable Splitting Approach

The depot may have an earliest departure time e_0 and latest arrival time u_0. We let T denote a scalar larger than the travel time of any feasible route. The formulation below is an extension of the formulations given for example by Solomon (1987).

Variables

$$x_{ijk} = \begin{cases} 1, & \text{if vehicle } k \text{ travels directly from customer } i \text{ to customer } j, \\ 0, & \text{otherwise.} \end{cases}$$

$$y_{ik} = \begin{cases} 1, & \text{if customer } i \text{ is visited by vehicle } k, \\ 0, & \text{otherwise.} \end{cases}$$

t_i = the time to begin delivery at customer i.

t_{0ek} = departure time of vehicle k from the depot.

t_{0uk} = arrival time of vehicle k at the depot.

Model

$$\min. \sum_{i \in N_0} \sum_{j \in N_0} \sum_{k \in M} c_{ij} x_{ijk} \tag{1}$$

$$\text{s.t.} \sum_{i \in N} x_{irk} - \sum_{j \in N} x_{rjk} = 0 \quad \forall\, r \in N,\ \forall\, k \in M, \tag{2}$$

$$\sum_{i \in N} x_{i0k} = \sum_{j \in N} x_{0jk} = 1 \quad \forall\, k \in M, \tag{3}$$

$$t_i + s_i + t_{ij} - (1 - x_{ijk})T \leqslant t_j \quad \forall\, i,j \in N,\ \forall\, k \in M, \tag{4}$$

$$t_{0ek} + t_{0j} - (1 - x_{0jk})T \leqslant t_j \quad \forall\, j \in N,\ \forall\, k \in M, \tag{5}$$

$$t_i + s_i + t_{i0} - (1 - x_{i0k})T \leqslant t_{0uk} \quad \forall\, i \in N,\ \forall\, k \subset M, \tag{6}$$

$$e_i \leqslant t_i \leqslant u_i \quad \forall\, i \in N, \tag{7}$$

$$e_0 \leqslant t_{0ek} \quad \forall\, k \in M, \tag{8}$$

$$t_{0uk} \leqslant u_0 \quad \forall\, k \in M, \tag{9}$$

$$\sum_{i \in N} \sum_{j \in N} q_i x_{ijk} \leqslant Q_k \quad \forall\, k \in M, \tag{10}$$

$$t_i \geqslant 0 \quad \forall\, i \in N_0, \tag{11}$$

$$x_{ijk} \in \{0, 1\} \quad \forall\, i, j \in N_0,\ \forall\, k \in M, \tag{12}$$

$$\sum_{k \in M} y_{ik} = 1 \quad \forall\, i \in N, \tag{13}$$

$$y_{ik} \in \{0, 1\} \quad \forall\, i \in N_0,\ \forall\, k \in M, \tag{14}$$

$$\sum_{j \in N} x_{ijk} = y_{ik} \quad \forall\, i \in N_0,\ \forall\, k \in M. \tag{15}$$

The objective function (1) minimizes the total travel cost. Constraints (2) ensure that if route k visits a point, it has to leave the point again (balance equation). Constraints (3) state that each route originates and terminates at the depot. Constraints (4)–(6) ensure compatible arrival times. Moreover, (4) eliminates all oriented subtours. (7) restricts the arrival time at a customer to its time windows. (8)–(9) are depot time window constraints on vehicle departure and arrival times. Constraints (10) ensure that the demand of each route is within the capacity limit of the vehicle serving the route. (13) ensures that each customer is visited exactly once. (15) constraints a route to leave a customer location exactly once. The constraints are arranged such that (1)–(12) are dealing only with variables x_{ijk} and t_i, while (13)–(14) are dealing only with variables y_{ik}. Constraint (15) shows the relations between the variables x_{ijk} and y_{ik}. A solution to the VRPTW provides an allocation of customers to vehicle (variables y_{ik}), the routes (variables x_{ijk}), and the vehicle schedules (variables t_i).

The first solution method chosen is based on the use of Lagrangian relaxation by splitting variables into multiple copies. The purpose is to create a problem that separates into a number of subproblems with known usable structures (Jörnsten et al. 1985, Guignard and Kim 1987). The formulation stated above is well suited for variable splitting by relaxing constraints (15) and introducing Lagrange multipliers λ_{ik}.

Subproblem 1 becomes

$$\left.\begin{array}{l} \min -\sum_i \sum_j \lambda_{ik} y_{ik} \\ \text{subject to (13) and (14)} \end{array}\right\}, \tag{16}$$

which is a semiassignment problem. The problem is easily solved by inspection.

Subproblem 2 becomes

$$\left.\begin{array}{l} \min \sum_i \sum_j (c_{ij} + \lambda_{ik}) x_{ijk} \\ \text{subject to (2)–(12)} \end{array}\right\}, \tag{17}$$

which is a shortest path problem with time windows and capacity constraints (SPTWCC). There are m such problems. The SPTWCC is NP-hard in the strong sense (Dror 1994) and only a relaxed version of the problem is solved. It is possible to generalize the GPLA algorithm (Desrochers and Soumis 1988) (used for the SPTW) to deal with the SPTWCC. In Desrocher et al. (1992), a similar approach is discussed.

The solution of the SPTWCC may contain negative cycles. In order to avoid too many solutions with negative cycles, two-cycle elimination (Kolen et al. 1987) is applied. A two-cycle is a cycle where a customer is visited twice with only one customer in between.

The coordinating problem arising from the relaxation of constraints (15) may be written in dual form as

$$w(\lambda^*) = \max w(\lambda), \quad \lambda \in R^{m(n+1)}, \tag{18}$$

where

$$w(\lambda) = \min\Big[\sum_i \sum_j c_{ij} \sum_k x_{ijk} + \sum_i \sum_k \lambda_{ik} \big(\sum_j x_{ijk} - y_{ik}\big) \Big].$$

The y's and x's are solutions to subproblems (16) and (17), respectively. Due to the integrality of x and y there may be a gap between the optimal values of the objective functions of the primal and the dual problem, respectively. In this case $w(\lambda^*)$ will only be a lower bound for the VRPTW.

Problem (18) is solved iteratively by subgradient optimization. To close the duality gap it was decided to introduce a branch-and-bound method and then use the variable splitting approach to determine lower bounds. The following separation rule was used in the branching process: Allocate a vehicle to a customer, i.e., fix one variable y_{ik} to one or zero. The chosen customer is the one which has been visited most often during the previous iterations if the vehicle capacity constraint allows it.

The subproblem SPTWCC has to be solved in such a way that the fixed vehicle-customer connections are satisfied. This is done by introducing an additional set of multipliers (penalties) connected to the vehicle-customer assignment. A large negative penalty encourages the assignment. A large positive penalty does the opposite.

1.2. The K-tree Approach

The K-tree approach is an extension of the classical 1-tree approach for the travelling salesman problem to the case with several capacity constrained vehicles. In this approach it is assumed that each route contains at least two customers. A special mathematical model is formulated and the constraints are Lagrangian relaxed. The approach can be extended to the case with time window constraints. Lagrangian relaxation is still applied. The relaxed problem is a degree constrained K-tree problem. The multipliers are determined using the subgradient method, and new capacity and time window constraints are generated as they are violated. For further details see Fisher (1994). It is expected, that the minimum K-tree method will perform most efficient on clustered problems with loose time windows, as fewer time window violating paths will occur, and fewer constraints are therefore expected to be generated.

2. COMPUTATIONAL RESULTS

A number of the Solomon's (1983) test problems has been solved. Note that for example problem *C*101 with 25 (50) nodes is the original problem *C*101 considering only the first 25 (50) customers. Problem *T*31 is a 31-customer problem. It is a reduced version of problem *RC*102. See (Madsen 1990) for further details. The variable splitting approach was run on a HP-9000/835 SE computer while the K-tree approach was run on an Apollo 3000 model 3010. The K-tree approach is using real valued distances while the variable splitting approach is using distances rounded to one digit. In the column generation approach (Desrochers et al. 1992), distances are truncated to one digit. This may result in small differences in the value of the optimal objective functions.

Table I presents the results from the variable splitting

Table I
Results from the Variable Splitting Approach

Problem	Nodes	Arcs	Lower Bound	Optimal Solution	Vehicles	B-B Nodes	Iterations	Gap %
*C*101	25	332	191.7	191.7	3	0	138	0
*C*101	50	1216	363.1	363.1	5	0	155	0
*C*101	100	4512	828.7	828.7	10	0	290	0
*C*102	25	473	190.6	190.6	3	0	170	0
*C*102	50	1730	362.0	362.0	5	0	156	0
*C*104	25	604	187.4	187.4	3	0	187	0
*C*105	25	357	191.7	191.7	3	0	77	0
*C*105	50	1332	363.1	363.1	5	0	197	0
*C*105	100	5049	828.7	828.7	10	0	344	0
*C*106	25	336	191.7	191.7	3	0	97	0
*C*106	50	1272	363.1	363.1	5	0	140	0
*C*106	100	5423	828.7	828.7	10	6	692	0
*C*107	25	377	191.7	191.7	3	0	78	0
*C*107	50	1443	363.1	363.1	5	0	145	0
*C*107	100	5616	828.7	828.7	10	9	720	0
*R*102	25	404	547.3	547.9	7	10	432	0.1
*R*104	25	559	417.3	418.0	4	3	326	0.2
*R*107	25	541	424.3	425.3	4	13	650	0.2
*R*109	25	411	442.7	442.7	5	0	160	0
*R*109	50	1539	766	"817"	8			6.7
*RC*101	25	277	419	461	4	1221	21674	10.0
*T*31	31	567	704.7	726.1	6	649	13233	3.0

Notes: "817" means feasible solution. Gap = (Optimal solution − Lower bound)/Lower bound in %. The Gap is before the Branch and Bound algorithm was applied. In *R*109 and *RC*101 the distances are rounded to nearest integer.

Table II
Results from the K-tree Approach

Problem	Nodes	Arcs	Lower Bound	Upper Bound	Vehicles	Gap (%)
*C*101	25	332	191.8136(+)	191.8136	3	0
	50	1216	363.2468(+)	363.2468	5	0
	100	4512	828.9369(+)	828.9369	10	0
*C*102	25	473	190.7376(+)	191.8136	3	0
	50	1730	362.1708(+)	363.2468	5	0.3
	100	6581	827.6304	859.7559	10	3.9
*C*103	25	569	186.9525	191.8136	3	2.7
	50	2165	360.0059	364.4184	5	1.2
*C*104	25	604	186.5071	187.8123	3	0.7
*C*105	25	357	191.8136(+)	191.8136	3	0
	50	1332	363.2468(+)	363.2468	5	0
*C*106	25	336	191.8136(+)	191.8136	3	0
	50	1270	363.2468(+)	363.2468	5	0
*C*107	25	377	191.8136(+)	191.8136	3	0
	50	1443	363.2468(+)	363.2468	5	0
	100	5616	828.9370(+)	828.9370	10	0
*C*108	25	431	187.5924	240.8674	3	28.4
*C*109	25	482	186.7668	198.7051	3	6.4
*R*101	25	224	619.1618	619.1719	8	"0"
	50	810	1042	1046.701	12	0.5
*R*102	25	404	530.0098	625.3718	7	18.0
	50	1437	862.7472	990.5188	11	14.8
*R*104	25	559	389.6417	475.4149	4	22.0
*R*105	25	299	512.3168	547.9680	6	7.0
*R*106	25	457	444.6233	512.5254	5	15.3
*R*108	25	585	382.5780	457.3669	4	19.6
*R*109	25	410	428.5840	592.1453	5	38.7
*R*112	25	646	376.3418	508.3654	4	35.1

Notes: (+) means that the K-tree solution is feasible. Gap = (Optimal solution − Lower bound)/Lower bound in %. "0"% means a very small number close to zero.

approach with no manual intervention. Twenty-one problems were solved to optimality, of which 16 showed no duality gap. Even though the results are preliminary, it seems that the variable splitting approach works best on *C*-problems while the approach is not yet capable to solve *RC*-problems. The CPU times for the C and R problems were between 3.5 and 151.9 secs (25 customers), between 46.6 and 289.4 secs (50 customers), between 616.0 and 4241.6 secs (100 customers). RC101 required 1358.1 secs, and T31 required 1765.5 secs.

Table II presents the results from the K-tree approach where some manual intervention was needed. Two 100-customer problems, five 50-customer problems, and five 25-customer problems were solved to optimality. In most of the *C*-problems the gap is rather small. In the *R*-problems the gap is larger and only smaller problems have been solved. No *RC*-problems have been solved.

3. CONCLUSION

In this paper we have presented two new optimization algorithms for the VRPTW: a generalized K-tree algorithm and a variable splitting algorithm. Both algorithms have been able to solve optimally problems with up to 100 customers. It is not yet possible to determine to which problem types the methods are best suited. However, it seems that they perform best on clustered problems. The two approaches cannot solve as many or as varied problems as the column generation approach (Desrochers et al. 1992). However, they offer insight into the VRPTW and along with the column generation method they are the only methods capable of optimally solving 100-customer problems.

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REFERENCES

CHRISTOFIDES, N., A. MINGOZZI, AND P. TOTH. 1981. State Space Relaxation Procedures for the Computation of Bounds to Routing Problems. *Networks* **11,** 145–164.

DESROCHERS, M., AND F. SOUMIS. 1988. A Generalized Permanent Labeling Algorithm for the Shortest Path Problems with Time Windows. *INFOR* **26,** 191–212.

DESROCHERS, M., J. K. LENSTRA, M. W. P. SAVELSBERGH, AND F. SOUMIS. 1988. Vehicle Routing with Time Windows: Optimization and Approximation. In *Vehicle Routing: Methods and Studies* by B. L. Golden and A. A. Assad (eds.). North Holland, Amsterdam, 65–84.

DESROCHERS, M., J. DESROSIERS, AND M. M. SOLOMON. 1992. A New Optimization Algorithm for the Vehicle Routing Problem with Time Windows. *Opns. Res.* **40,** 342–354.

DROR, M. 1994. Note on the Complexity of the Shortest Path Models for Column Generation in VRPTW. *Opns. Res.* **42,** 977–978.

FISHER, M. L. 1994. Optimal Solution of Vehicle Routing Problems Using Minimum K-Trees. *Opns. Res.* **42,** 626–642.

GUIGNARD, M., AND S. KIM. 1987. Lagrangean Decomposition: A Model Yielding Stronger Lagrangean Bounds. *Math. Program.* **39,** 215–228.

HALSE, K. 1992. Modeling and Solving Complex Vehicle Routing Problems. Ph.D. Dissertation. IMSOR, The Technical University of Denmark, Lyngby, Denmark.

JÖRNSTEN, K., M. NÄSBERG, AND P. SMEDS. 1985. Variable Splitting—A New Lagrangean Approach to Some Mathematical Programming Models. Report LITH-MAT-85-04, Department of Mathematics, Linköping Institute of Technology, Linköping, Sweden.

JÖRNSTEN, K., O. B. G. MADSEN, AND B. SØRENSEN. 1986. Exact Solution of the Vehicle Routing and Scheduling Problem with Time Windows by Variable Splitting. Research Report 5/1986. IMSOR, The Technical University of Denmark, Lyngby, Denmark.

KOLEN, A. W. J., A. H. G. RINNOOY KAN AND H. W. J. M. TRIENEKENS. 1987. Vehicle Routing with Time Windows. *Opns. Res.* **35,** 266–273.

MADSEN, O. B. G. 1988. Variable Splitting and Vehicle Routing Problems with Time Windows. Preprint IB/1988. IMSOR, The Technical University of Denmark, Lyngby, Denmark.

MADSEN, O. B. G. 1990. Lagrangean Relaxation and Vehicle Routing. Working Paper. IMSOR, The Technical University of Denmark, Lyngby, Denmark.

SOLOMON, M. M. 1983. Vehicle Routing and Scheduling with Time Window Constraints: Models and Algorithms. Ph.D. Dissertation, Department of Decision Sciences, University of Pennsylvania, Philadelphia.

SOLOMON, M. M. 1987. Algorithms for Vehicle Routing and Scheduling Problems with Time Window Constraints. *Opns. Res.* **35,** 254–265.

SOLOMON, M. M., AND J. DESROSIERS. 1988. Time Window Constrained Routing and Scheduling Problems. *Trans. Sci.* **22,** 1–13.