Appendix B. A Brief Review of Copulas Functions

Copulas functions are a statistical tool which has many advantages. First, copulas make it possible to determine the nature of dependence of the series, be it linear or not, monotone or not. In addition to the fact that they over a great flexibility in the implementation of the multivariate analysis, copula authorizes a wider selection of the marginal distributions of the financial series. Second, they allow a less banal representation of the statistical dependence in finance based on the traditional correlation measure (Embrechts et al., 1997). Third, they authorize less restrictive univariate probability distributions which make it possible to better accounting for the stylized facts in finance (leptokurtosis, asymmetry, tail dependence). Fourth, they consider very general multivariate distributions, independently of the laws of the marginal ones which can have different laws and be unspecified. Furthermore, the copulas approach enables us to ease the implementation of multivariate models. Indeed, this approach allows the decomposition of the multidimensional law into its univariate marginal functions and a dependence function that would make possible extensions of some results obtained in the univariate case to the multivariate case. Hence, copula is an exhaustive statistic of the dependence. Finally, Patton (2006a) shows that copulas are useful extensions and generalizations of approaches for modeling joint distributions that have appeared in the literature.[1]

Specifically, a copula, defined by Sklar (1959), is a function that links together univariate distribution functions to form a multivariate distribution function. If all of the variables are continuously distributed, then their copula is simply a multivariate distribution function with uniform (0, 1) univariate marginal distributions. Copulas have been used both in multivariate time series analysis, where they are used to characterize the (conditional) cross-sectional dependence between individual time series, and in univariate time series analysis, and are used to characterize the dependence between a sequences of observations of a scalar time series process.

In the following, we refer Patton's (2006b) work to illustrate the concepts of copula. Consider a vector random variable, $\mathbf{X} = [X_1, X_2, ..., X_n]'$, with joint distribution $\mathbf{F}$ and marginal distributions $F_1, F_2, ..., F_n$. Sklar's (1959) theorem provides the

[1] The number of papers on copula theory in finance and economics has grown enormously in recent years. One of the most influential of the early papers on copulas in finance is that of Embrechts, McNeil and Straumann (2002). Since then, scores of papers have been proposed, exploring the uses of copulas in finance, macroeconomics, and microeconomics, as well as developing the estimation and evaluation theory required for these applications. Joe (1997) and Nelsen (2006) provide detailed and readable introductions to copulas and their statistical and mathematical foundations, while Ghysels, Gouriéroux, and Jasiak (2004) focus primarily on applications of copulas in mathematical finance and derivatives pricing.

mapping from the individual distribution functions to the joint distribution function:

$$\mathbf{F}(\mathbf{x}) = \mathbf{C}\left[F_1(x_1), F_2(x_2), ..., F_n(x_n)\right], \quad \forall \mathbf{x} \in \mathbb{R}^n \tag{1}$$

From any multivariate distribution, $\mathbf{F}$, we can extract the marginal distributions, F_i, and the copula, $\mathbf{C}$. And, more useful for time series modeling, given any set of marginal distributions $(F_1, F_2, ..., F_n)$ and any copula $\mathbf{C}$; Eq. (1) can be used to obtain a joint distribution with the given marginal distributions. An important feature of this result is that the marginal distributions do not need to be in any way similar to each other, nor is the choice of copula constrained by the choice of marginal distributions. This flexibility makes copulas a potentially useful tool for building econometric models.

Since each marginal distribution, F_i contains all of the univariate information on the individual variable X_i, while the joint distribution $\mathbf{F}$ contains all univariate and multivariate information, it is clear that the information contained in the copula $\mathbf{C}$ must be all of the dependence information between the X_i's. It is for this reason that copulas are sometimes known as 'dependence functions', see Galambos (1978). If the joint distribution function is n-times differentiable, then taking the nth cross-partial derivative of Eq. (1) we obtain,

$$\begin{aligned} \mathbf{f}(\mathbf{x}) &\equiv \frac{\partial^n}{\partial x_1 \partial x_1 \cdots \partial x_n} \mathbf{F}(\mathbf{x}) \\ &= \prod_{i=1}^{n} f_i(x_i) \cdot \frac{\partial^n}{\partial x_1 \partial x_1 \cdots \partial x_n} \mathbf{C}[F_1(x_1), F_2(x_2), ..., F_n(x_n)] \\ &\equiv \prod_{i=1}^{n} f_i(x_i) \mathbf{c}[F_1(x_1), F_2(x_2), ..., F_n(x_n)] \end{aligned} \tag{2}$$

Thus, the joint density is equal to the product of the marginal densities and the 'copula density', denoted by $\mathbf{c}$. This of course also implies that the joint log-likelihood is simply the sum of univariate log-likelihoods and the .copula log-likelihood., which is useful in the estimation of copula-based models:

$$\log \mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} \log f_i(x_i) + \log \mathbf{c}\left[F_1(x_1), F_2(x_2), ..., F_n(x_n)\right], \tag{3}$$

The decomposition of a joint distribution into its marginal distributions and copula allows the researcher a great deal of flexibility in specifying a model for the joint distribution. This is clearly an advantage when the shape and goodness-of-fit of the model for the joint distribution is of primary interest. In situations where the researcher has accumulated knowledge about the distributions of the individual variables and wants to use that in constructing a joint distribution, copulas also have a valuable role.

Figure B shows the scatter plots of simulated bivariate copulas: Gumbel, which is a special case of the Clayton–Gumbel copula that exhibits only upper tail dependence (left-top panel), Clayton, which is a special case of the Clayton–Gumbel copula that exhibits only lower tail dependence (right-top panel), Frank (left-bottom panel) and Student (right-bottom panel).

Figure B. Simulations of Copulas Functions

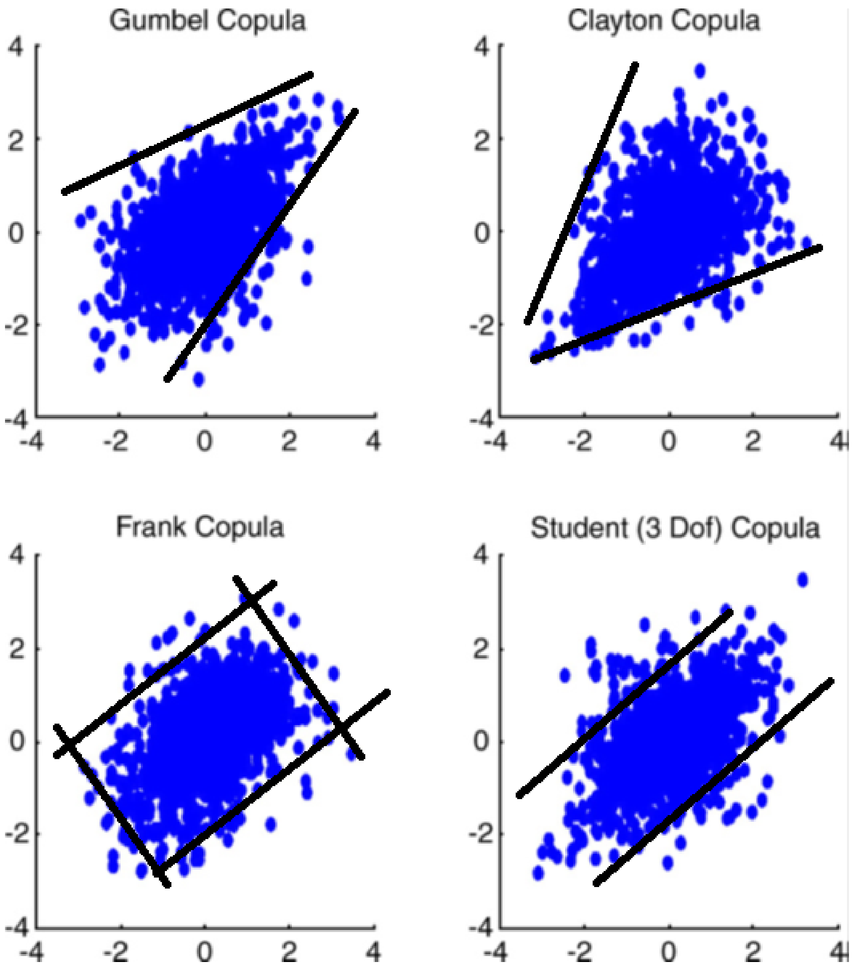

Fig. 3. Simulated Copulas scatter plots of simulated Gumbel, Clayton, Frank, and Student Copulas. All marginal are standardized normals. The parameters of the copulas were chosen to give a Kendall's tau equal to 0.3.

In all cases, 1000 observations were generated, and margins were selected as standard normal. The parameters of the copulas were chosen to give a Kendall's tau equal to 0.3. Therefore, the simulated random variables in Fig. 3 differ only on the dependence structure, with the Clayton copula showing strong association in the left tail, while the Gumbel copula shows strong association in the right tail. It is in this sense that the Clayton and Gumbel copulas describe asymmetric dependence. On the other hand, the Student copula exhibits dependence in both tails, while no clear association in the tails can be observed for the Frank copula.