Appendix A. The Concepts of Copula

In the following, we refer Patton's (2006b) work to illustrate the concepts of copula. Consider a vector random variable, $\mathbf{X} = [X_1, X_2, ..., X_n]'$, with joint distribution $\mathbf{F}$ and marginal distributions $F_1, F_2, ..., F_n$. Sklar's (1959) theorem provides the mapping from the individual distribution functions to the joint distribution function:

$$\mathbf{F}(\mathbf{x}) = \mathbf{C}\left[F_1(x_1), F_2(x_2), ..., F_n(x_n)\right], \quad \forall \mathbf{x} \in \mathbb{R}^n \tag{A1}$$

From any multivariate distribution, $\mathbf{F}$, we can extract the marginal distributions, F_i, and the copula, $\mathbf{C}$. And, more useful for time series modeling, given any set of marginal distributions $(F_1, F_2, ..., F_n)$ and any copula $\mathbf{C}$; Eq. (1) can be used to obtain a joint distribution with the given marginal distributions. An important feature of this result is that the marginal distributions do not need to be in any way similar to each other, nor is the choice of copula constrained by the choice of marginal distributions. This flexibility makes copulas a potentially useful tool for building econometric models.

Since each marginal distribution, F_i contains all of the univariate information on the individual variable X_i, while the joint distribution $\mathbf{F}$ contains all univariate and multivariate information, it is clear that the information contained in the copula $\mathbf{C}$ must be all of the dependence information between the X_i's. It is for this reason that copulas are sometimes known as 'dependence functions', see Galambos (1978). If the joint distribution function is n-times differentiable, then taking the nth cross-partial derivative of Eq. (A1) we obtain,

$$\begin{aligned} \mathbf{f}(\mathbf{x}) &\equiv \frac{\partial^n}{\partial x_1 \partial x_1 \cdots \partial x_n} \mathbf{F}(\mathbf{x}) \\ &= \prod_{i=1}^{n} f_i(x_i) \cdot \frac{\partial^n}{\partial x_1 \partial x_1 \cdots \partial x_n} \mathbf{C}\left[F_1(x_1), F_2(x_2), ..., F_n(x_n)\right] \quad , \\ &\equiv \prod_{i=1}^{n} f_i(x_i) \mathbf{c}\left[F_1(x_1), F_2(x_2), ..., F_n(x_n)\right] \end{aligned} \tag{A2}$$

Thus, the joint density is equal to the product of the marginal densities and the 'copula density', denoted by $\mathbf{c}$. This of course also implies that the joint log-likelihood is simply the sum of univariate log-likelihoods and the .copula log-likelihood., which is useful in the estimation of copula-based models:

$$\log \mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} \log f_i(x_i) + \log \mathbf{c}\left[F_1(x_1), F_2(x_2), ..., F_n(x_n)\right], \tag{A3}$$

The decomposition of a joint distribution into its marginal distributions and copula allows the researcher a great deal of flexibility in specifying a model for the joint distribution. This is clearly an advantage when the shape and goodness-of-fit of the model for the joint distribution is of primary interest. In situations where the researcher has accumulated knowledge about the distributions of the individual variables and wants to use that in constructing a joint distribution, copulas also have a valuable role. In other situations, for example when the researcher is primarily focused on the conditional mean and/or conditional variance of a vector of variables, copulas may not be the 'right tool for the job', and more standard vector autoregressive models and/or multivariate GARCH models, see Silvennoinen and Teräsvirta (2009), may be more appropriate. For a lively discussion of the value of copulas in statistical modeling of dependence, see Joe (1997) and Nelson (2006) and Mikosch (2006) for more details. Different copulas usually represent different dependence structures with the so-called association parameter, θ_c, which indicates the strength of the dependence. Some commonly used copulas in economics and finance include: the bivariate Gaussian copula, the student-*t* copula, the Gumbel copula, the Clayton copula, and their combinations. The Gaussian copula does not have tail dependence, while the *t* copula has symmetric tail dependence; and the Gumbel copula has only upper tail dependence, while the Clayton copula has lower tail dependence.

Reference

Galambos, J., 1978, *The Asymptotic Theory of Extreme Order Statistics*, Wiley, New York.

Joe, H., 1997. *Multivariate Models and Dependence Concepts*. Chapman and Hall.

Mikosch, T., 2006. Copulas: Tales and Facts, with Discussion and Rejoinder. *Extremes* 9, 3–62.

Nelsen, R., 2006. *An Introduction to Copulas*, Second Edition, Springer, U.S.A.

Patton, A., 2006b. Estimation of Multivariate Models for Time Series of Possibly Different Lengths. *Journal of Applied Econometrics* 21, 147–173.

Silvennoinen A. and T. Teräsvirta, 2009. Modelling Multivariate Autoregressive

Conditional Heteroskedasticity with the Double Smooth Transition Conditional Correlation GARCH Model. *Journal of Financial Econometrics* 7, 373–411.

Sklar, A., 1959. Fonctions De Répartition à *n* Dimensions et Leurs Marges. *Publications de l' Institut Statistique de l'Universite' de Paris* 8, 229–231.