Abstract—In this paper, the interaction equations for a flexible beam with multiple-bonded piezoelectric films are derived. Any flexible structure composed of flexible beam members can be analyzed with these equations. The interaction equations account for the effects of shaped piezoelectric films, that is, the film's electrodes may be etched with geometrical shapes and the poling direction may be altered by splicing. The derivation is based on a generalization of the interaction equations for a single film. The interaction equations make use of the vibrational modes of a structure. The vibrational modes can be derived either analytically or numerically using finite-element techniques. Experimental validation has been done on a cantilever beam with bonded polyvinylidene fluoride and lead zirconium titanate films. These equations are useful for developing techniques for shape control and active vibration damping in flexible structures.

Index Terms—Dynamic modeling, piezoceramic, piezoelectric, polyvinylidene fluoride, smart structures, spatial filtering.

I. INTRODUCTION

In the next decade, it is envisaged that large flexible space structures (LFSS’s) will be used in a variety of applications. Some potential applications include the space-based radar, the mobile servicing subsystem, and the space station. Due to the light weight and expansive sizes of LFSS’s, these structures will have significant structural flexibility. The elastic body modes will be very lightly damped and range from very low to very high frequencies. High-frequency modes can be attenuated by using materials with significant viscoelasticity, whereas low-frequency modes cannot. It is the low-frequency modes that will limit the performance of a control system for an LFSS.

Structural vibrations can be actively damped using piezoelectric films that are bonded to structural members of a flexible structure. The dual nature of piezoelectric films allows them to be used as either actuators or sensors. Shaping of piezoelectric films, by splicing and electrode etching, can be used to sense and control individual modes in a structure. This enables a feedback controller to be realized in terms of a suitably shaped film. In principle, the feedback can be infinite order, as it is only limited by the fidelity with which the electrode can be etched [1]. Traditionally, controller design has been done using a temporal paradigm where controller complexity is measured in terms of dynamic order. Piezoelectric films allow control design to be done using a spatial paradigm [2], [3], where controller complexity is measured in terms of film size and electrode geometry. This new control paradigm will enable the development of new control design methodologies. A precursor to this work is the development of interaction equations for modeling the dynamics of flexible structures with piezoelectric films.

Some researchers have developed a finite-element model (FEM) to represent the composite dynamics of a flexible substructure with bonded piezoelectric films. The disadvantage of this method is that each different placement and shaping of the piezoelectric films requires that the FEM be recomputed. This makes it awkward to use film shape and placement as a design parameter. A complex computational algorithm must be used to deal with the resultant coupling of controller design with structural modeling [4].

Crawley and de Luis [5] developed a useful two-step procedure for analyzing the coupling of piezoelectric film actuators and a flexible substructure. First, a dynamic model of a structure without piezoelectric films is developed using a conventional technique, analytical solution of a partial differential equation, or numerical solution of a finite-elements problem. Second, the influence of piezoelectric film upon the substructure is derived in terms of generalized force from a static stress–strain model of the piezoelectric film. This quasi-static approach works well under the assumptions that actuator mass is small compared to total mass of the structure, and the actuator resonance frequencies are much higher than frequencies of interest. This method has the advantage that different film placements and sizes can be studied without recomputation of the substructure dynamics.

Lee and Moon [6] studied the interactions between a piezoelectric film and a flexible plate. The interactions are expressed in terms of eigenvalues to the partial differential equation of plate vibration. Accordingly, a piezoelectric film is characterized as a series of modal forces, when used as an actuator, and as a series of modal charges, when used as a sensor. The orthogonal nature of the mode shapes can be used to construct modal actuators/sensors which excite/observe particular vibrational modes.
In this paper, the modal techniques of Lee and Moon [6] are combined with the quasi-static technique of Crawley and de Luis [5] to yield a novel set of interaction equations. These equations are used to characterize the operation of shaped piezoelectric films that can be used as either actuators or sensors. These equations are used to derive a state-space model of a flexible structure with multiple bonded piezoelectric films. This model is useful for the design of control techniques for implementing active vibration damping (see [7]).

This paper is organized as follows. In Section II, electro-mechanical models of piezoelectric films are described. The operation of shaped piezoelectric films are discussed in Section III. The static stress and strain equilibrium conditions for a flexible beam with bonded piezoelectric films are derived in Section IV. The piezoelectric film actuator equations are derived in Section V. The piezoelectric film sensor equations are derived in Section VI. Experimental results are discussed in Section VII. Concluding remarks are made in Section VIII.

II. MODELING PIEZOELECTRIC FILMS

Consider the rectangular piezoelectric film illustrated in Fig. 1. The 1 axis and 2 axis define a plane which is parallel to the film’s surface. The 3 axis is perpendicular to the film’s surface. The 3 axis points in the opposite direction to the electric field used to pole the piezoelectric film during its manufacture. Some piezoelectric films, such as polyvinylidene fluoride (PVDF), are uniaxially stretched during the poling process. In such a case, the 1 axis is oriented in the direction of the uniaxial stretching. Piezoelectric films that are uniaxially stretched exhibit properties that differ in the directions of the 1 and 2 axes (e.g., \( d_{31} \neq d_{32} \)). Films that are not uniaxially stretched, such as lead zirconium titanates (PZT’s), do not exhibit these differences.

The transverse effects in piezoelectric films are illustrated in Fig. 1. The film is assumed to have a length \( L_e \) along the 1 axis, a width \( W_e \) along the 2 axis, and a thickness \( t_e \) along the 3 axis. A voltage \( V_3 \) is measured across the upper and lower electrodes. The upper and lower electrodes are assumed to have free charges of \( +Q \) and \( -Q \), respectively. A tensile force \( F_1 \) and an elongation displacement \( Y_1 \) are measured along the 1 axis. If the film acts as an actuator, a voltage \( V_3 \) is applied across the film; the induced polarization in the piezoelectric generates a tensile force \( F_1 \) and an elongation \( Y_1 \). If the film is used as a sensor, a tensile force \( F_1 \) is applied which stretches the film by \( Y_1 \); the induced polarization in the piezoelectric causes an increase of free charges on the contacts, which generates a voltage \( V_3 \) through a capacitive effect.

A piezoelectric film’s behavior can be modeled in terms of force and voltage [8]

\[
Y_1 = s_f^F F_1 + d_f V_3 \\
Q = d_f F_1 + c_f^F V_3
\]

(1)

where \( s_f^F \) is interpreted as the film compliance at constant electric field intensity; \( d_f \) the film charge to force ratio; and \( c_f^F \), the film capacitance at constant stress. The above film parameters are defined as follows:

\[
s_f^F := \frac{L_e}{E_e W_e t_e} , \quad d_f := \frac{d_{31} L_e}{t_e} , \quad c_f^F := \frac{\varepsilon_e L_e W_e}{t_e}
\]

where \( E_e \) is the Young's modulus of the piezoelectric at constant field intensity, \( \varepsilon_e \) is the permittivity of the piezoelectric at a constant stress, and \( d_{31} \) is the transverse piezoelectric charge to stress ratio. Equation (1) can be represented in terms of the electromechanical models shown in Fig. 2. Due to the inverted voltage source in the circuit of Fig. 2(b), an external applied force causes a negative piezoelectric displacement in the mechanical model in Fig. 2(c); thus, the mechanical stiffness of the film is augmented by the piezoelectric effect.

Piezoelectric films can also be modeled in terms of voltage and displacement [8]

\[
F_1 = \left( \frac{1}{s_f^F} \right) Y_1 - e_f V_3 \\
Q = e_f Y_1 + c_f^F V_3
\]

(2)
where \( c_f^S \) is interpreted as the film capacitance at constant strain, and \( e_f \) is the film charge to displacement ratio. The above parameters are defined as follows:

\[
\begin{align*}
 c_f^S &= c_f^S - \frac{d^2}{s_f^2} = (e_c - d_{33} E_c) \left( \frac{L_c W_c}{t_c} \right) \\
 e_f &= \frac{d_f}{s_f^2} = d_{33} E_c W_c.
\end{align*}
\]

Equation (2) can be represented in terms of the electromechanical models shown in Fig. 3.

**III. Shaped Piezoelectric Films**

A shaped film is idealized as a rectangular piezoelectric film that has a poling direction that varies throughout the film, and electrodes that are etched into geometrical shapes. At any point in a shaped film, the poling direction may either point up or down. The electrode shapes are such that a point in the piezoelectric film may or may not be covered by both electrodes. The properties of a shaped film are determined by the pattern of the poling direction variation and the electrode geometry. A shaped film can be constructed from several electrically connected contiguous sections of piezoelectric films. The orientation of the poling direction varies between sections. This construction is illustrated in Fig. 4.

Shaped films are analyzed in terms of variations of the poling direction and domain of the outer electrodes. The functions used to represent these quantities are expressed in terms of the coordinate system of the beam to which they are attached. The \( x, y, z \) coordinate frame has its origin on the neutral surface of the beam. The beam has a length \( L \) along the \( x \) axis, a width \( W_b \) along the \( y \) axis, and a thickness \( t_b \) along the \( z \) axis. The upper and lower surfaces of the beam have \( z \) coordinates of \( +t_b/2 \) and \( -t_b/2 \), respectively.

The shaped film is bonded to the beam as shown in Fig. 5. The film is situated between \( x = a - m \) and \( x = a + m \) in the \( x \) axis and \( y = 0 \) and \( y = W_c \). The \( z \) coordinate of the beam surface, to which the film is bonded, is denoted by \( z_c \). If the film is bonded to the top surface, \( z_c = t_b/2 \); otherwise, \( z_c = -t_b/2 \) for the lower surface. The film is assumed to be rectangular, but the electrode is assumed to have a general shape.

The bending strain of the flat beam varies only in the \( x \) and \( z \) directions, and time \( t \). Hence, beam strain \( e_b \) can be represented as a function \( e_b(x, z, t) \). Note the strains have the following antisymmetric property: \( e_b(x, -z, t) = -e_b(x, z, t) \). Accordingly, the strain in a cross section of the piezoelectric film is represented by a function \( e_c(x, z_c, t) \). The strain variations of \( e_c \) in the \( z \) direction are negligible due to the thinness of the piezoelectric film. The piezoelectric film is assumed to be perfectly bonded to the beam; hence, \( e_c(x, z_c, t) = e_b(x, z_c, t) \). In practice, the errors incurred by this simplifying assumption are typically negligible.

The properties of a shaped film are characterized in terms of domain function \( F(x, y) \) and poling function \( P(x, y) \). The domain function \( F(x, y) \) defines the region of a shaped film’s etched electrode: \( F(x, y) = 1 \), if point \((x, y)\) is covered by both electrodes; \( F(x, y) = 0 \), otherwise. The poling direction is specified with respect to the electric field intensity associated with a particular electrode polarity. The poling function \( P(x, y) \) specifies the variations of the poling direction throughout a shaped film, \( P(x, y) = +1 \), if poling direction at \((x, y)\) points toward negative electrode; \( P(x, y) = 0 \), both
film electrodes do not cover point \((x, y)\); \(P(x, y) = -1\), if poling direction at \((x, y)\) points toward positive electrode. These two functions are fundamental to the understanding of interactions between piezoelectric films and flexible structures.

The \textit{shape function} \(\Xi(x)\) of a shaped film is the average of the poling function \(P(x, y)\) over a cross section with a fixed \(x\) coordinate. It is defined by

\[
\Xi(x) := \frac{1}{W_c} \int_0^{W_c} P(x, y) \, dy.
\]

\(\Xi(x)\) succinctly expresses the key property that determines the operation of piezoelectric film sensors and actuators.

The \textit{area covered by the film’s electrodes and capacitance of the film} are given by

\[
A_c = \int_{a-m}^{a+m} \int_0^{W_c} F(x, y) \, dy \, dx \quad \text{and} \quad C_c = \frac{\varepsilon_c A_c}{t_c}
\]

where \(\varepsilon_c\) is the permittivity of the piezoelectric, and \(t_c\) is the \textit{film thickness}. If a voltage \(V(t)\) is impressed across the film’s electrodes, the \textit{piezoelectric strain} \(\Lambda(x, y, t)\) at a point \((x, y)\) is given by

\[
\Lambda(x, y, t) = \frac{d_{33}}{t_c} P(x, y)V(t).
\]

This follows upon applying (1) and the mechanical model of Fig. 2(c) to an infinitesimal piezoelectric film element. The \textit{average strain} over the cross section with a fixed \(x\) coordinate is \(\Lambda_a(x, t)\). It is defined by

\[
\Lambda_a(x, t) := \frac{d_{33}}{t_c} \Xi(x)V(t).
\]

This expression is used in the next section to determine the forces generated by a piezoelectric actuator.

When a shaped film is strained, an electric flux density \(D_F(x, y, t) = d_{33} E_c P(x, y) \varepsilon_c(x, z_c, t)\) is created due to the polarization in the piezoelectric. This follows upon applying (1) and the circuit model of Fig. 3(a) to an infinitesimal element of a piezoelectric film. The charge \(Q_c(t)\) corresponding to the charge source in the circuit of Fig. 3(a) is given by

\[
Q_c(t) = \int_{a-m}^{a+m} \int_0^{W_c} D_F(x, y, t) \, dx \, dy = d_{33} E_c W_c \int_{a-m}^{a+m} \Xi(x) \varepsilon_c(x, z_c, t) \, dx.
\]

For a given strain profile \(\varepsilon_c(x, z_c, t)\), the corresponding open-circuit voltage \(V_{oc}(t)\) and the short-circuit current \(I_{sc}(t)\) are, respectively, given by

\[
V_{oc}(t) = -\frac{d_{33} E_c W_c}{C_c} \int_{a-m}^{a+m} \Xi(x) \varepsilon_c(x, z_c, t) \, dx \quad \text{and} \quad I_{sc}(t) = -\frac{d_{33} E_c W_c}{C_c} \int_{a-m}^{a+m} \Xi(x) \frac{\partial \varepsilon_c(x, z_c, t)}{\partial t} \, dx.
\]

IV. STATIC STRESS AND STRAIN EQUILIBRIUM CONDITIONS

In this section, the static relationships between stress and strain of a flexible beam and bonded shaped piezoelectric films are derived [9], [10]. Consider a slice of the beam and a bonded piezoelectric film, as shown in Fig. 6. The bonding is assumed to be perfect so that the surface strain of the beam is assumed to equal the strain of the bonded piezoelectric film. The corresponding strain profile is shown in Fig. 6. The strain profile corresponds to the application of a positive moment that causes the top of the beam and piezoelectric film to expand under tension, and the bottom of the beam to contract under compression.

The relationship between piezoelectric film strain to induced beam strain can be derived using elastic strain energy with the principle of virtual work. Strain energy density \(u_c\) in a piezoelectric film is given by the integral [11]

\[
u_c(x, y, z_c) = \int_0^{z_c} \sigma_c(x, y, z_c) \, dz
\]

where \(\sigma_c(x, y, z_c) = E_c (\varepsilon_c(x, y) - \Lambda_c(x, y))\) is the stress as a function of strain \(\varepsilon\) in the piezoelectric film. The strain energy in an infinitesimal slice of piezoelectric film of length \(\Delta x\) is given by

\[
W_c \Delta x = \left( \int_0^{W_c} \int_0^{t_c} u_c \, dy \, dz \right) \Delta x
\]

The strain energy in a slice of length \(\Delta x\) of a bent beam [11] with a surface strain \(\varepsilon_b(x, z_c)\) is

\[
U_b \Delta x = \frac{2E_b I_b}{t_b^3} \varepsilon_b(x, z_c) \Delta x
\]

The principle of virtual work [11] can be used to determine the static interaction between the piezoelectric film and the beam. Since the beam and the piezoelectric film are assumed to be in equilibrium,

\[
\delta U_b \Delta x + \delta U_c \Delta x = -\frac{4E_b I_b}{t_b^3} \varepsilon_b(x, z_c) \Delta x \cdot \delta \varepsilon_b(x, z_c) + W_c E_c \Lambda_c \varepsilon_c(x, z_c) - \Lambda_a \Delta x \cdot \delta \varepsilon_c(x, z_c) = 0,
\]
Using the perfect bonding assumption, \( \varepsilon_c(x, z_c) = \varepsilon_u(x, z_c) \) and \( \delta \varepsilon_c(x, z_c) = \delta \varepsilon_u(x, z_c) \), yields
\[
\varepsilon_u(x, z_c) = \left( \frac{W_c E_c}{W_c E_c + 4E_b t_b \varepsilon_c} \right) \Lambda_0(x).
\]  
(5)

A flat beam has a cross-sectional moment of inertia \( I_b = W_b t_b^3/12 \). Substituting into (5) yields
\[
\varepsilon_c(x, z_c) = \varepsilon_u(x, z_c) = \left( \frac{3W_c}{W_b} \right) \alpha \left( \frac{3W_c}{W_b} + \frac{E_b t_b}{E_c t_c} \right) \Lambda_0(x).
\]
\[
= \frac{\alpha}{\alpha + \psi} \Lambda_0(x)
\]  
(6)

where \( \alpha = 3W_c/W_b \) and \( \psi = E_b t_b/E_c t_c \). Equation (6) relates piezoelectric strain to induced beam strain. This relationship can be used to derive the effective distributed moment and effective distributed force that is induced by the piezoelectric film. From beam theory \( [11] \), a surface strain of \( \varepsilon_u(x, z_c) \) corresponds to a distributed moment \( M_{eff} \) given by
\[
M_{eff}(x) = -\left( \frac{2E_b t_b}{I_b} \right) \varepsilon_u(x, z_c)
= -\frac{E_b t_b W_c}{2(\alpha + \psi)} \Lambda_0(x).
\]  
(7)

A distributed moment \( M_{eff} \) is equivalent to a distributed upward \( F_{eff} \) force given by
\[
F_{eff} = \frac{d^2 M_{eff}(x)}{dx^2} = \frac{E_b t_b^2 W_c}{2(\alpha + \psi)} \times \frac{d^2 \Lambda_0(x)}{dx^2}.
\]
(8)

The stress–strain relationships derived by Crawley and deLuis \( [5] \) can be derived by using the above derivation with a uniform shape function, that is, \( \Xi(x) = 1 \). Equations (6)–(8) have the advantage that they have a simple form that does not involve the boundary conditions derived from the strains at the ends of the piezoelectric film. Accordingly, (6)–(8) are simpler to use than those of Crawley and deLuis \( [5] \).

V. DYNAMIC ACTUATOR EQUATIONS

In this section, the equations that determine the dynamic influence of a piezoelectric film actuator upon a flexible substructure are derived. The interaction equations are derived using the assumed modes method (AMM) \( [12] \). The use of this analytical technique is predicated on the assumption that the actuator does not significantly affect the mode shape of the flexible substructure. This is a reasonable assumption for piezoelectric films bonded to flexible structural elements.

The response of the flexible structure can be modeled in terms of a series \( u(x, t) = \sum_{i=1}^{n} \Phi_i(x) q_i(t) \), where \( u(x, t) \) is displacement, \( \Phi_i(x) \) is the \( i \)th mode shape, and \( q_i(t) \) is the \( i \)th modal amplitude. This series can be derived analytically for simple structures, and by FEM for complex structures. The kinetic energy of the system can be written as
\[
T(t) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} q_i(t) q_j(t)
\]
where the symmetric mass coefficients \( m_{ij} \) depend on the continuous system mass properties and the functions \( \Phi_i(x) \). The potential energy can be written as
\[
V(t) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} q_i(t) q_j(t)
\]
where the symmetric stiffness coefficients \( k_{ij} \) depend on the continuous system stiffness properties and the derivatives of \( \Phi_i(x) \). External forces are nonconservative, so that the Lagrange’s equations of motion have the form
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} = \dot{Q}_r(t), \quad \text{for } r = 1, 2, \ldots, n
\]
where \( \dot{Q}_r(t) \) is a generalized nonconservative force. Inserting \( T(t) \) and \( V(t) \) into Lagrange’s equation of motion yields
\[
\sum_{i=1}^{n} m_{ri} \ddot{q}_i(t) + \sum_{i=1}^{n} \dot{k}_{ri} q_i(t) = \dot{Q}_r(t), \quad \text{for } r = 1, 2, \ldots, n.
\]

The dynamic interactions between a shaped piezoelectric film actuator and a beam can be most conveniently derived by adding the strain energy of the piezoelectric film \( U_c \) to the potential energy function \( V(t) \) for a flexible beam. This results in a composite potential energy function \( V^c(t) \). This technique is adapted from de Luis et al. \( [13] \). This composite energy function is used in conjunction with the AMM to derive the dynamic equations.

Consider the beam with the bonded piezoelectric film shown in Fig. 5. The strain of the piezoelectric film \( \varepsilon_c(x, z_c) \) is derived from the surface strain of the beam \( \varepsilon_u(x, z_c) \) as follows:
\[
\varepsilon_c(x, z_c) = \varepsilon_u(x, z_c) = -\frac{b \partial u(x, t)}{2 \partial x^2} = -\frac{b}{2} \left[ \sum_{i=1}^{n} \frac{\partial \Phi_i(x)}{\partial x^2} q_i(t) \right].
\]

Piezoelectric strain energy \( U_c \) is obtained by integrating \( U_c^c(x) \) along the length of the film
\[
U_c = W_c E_c t_c \left\{ \int_{a_{-m}}^{a_{+m}} \left[ \sum_{i=1}^{n} \frac{\partial \Phi_i(x)}{\partial x^2} q_i(t) \right] \right\} dx + \int_{a_{-m}}^{a_{+m}} \Lambda_0 \left\{ \sum_{i=1}^{n} \frac{\partial \Phi_i(x)}{\partial x^2} q_i(t) \right\} dx.
\]

Using the composite potential energy function \( V^c(t) = U_c + V(t) \) in Lagrange’s equations of motion yields the following
set of differential equations:

\[ \ddot{q}_i(t) + \sum_{j=1}^{n} (k_{ij} + K_{ij}^{\text{PZEO}}) \dot{q}_j(t) = F_i(t), \quad i = 1, 2, 3, \ldots, n \]  

(9)

where \( k_{ij} := E_t b_i \lambda_{ij}^2 / \rho \) for \( i = j \), \( k_{ij} := 0 \) for \( i \neq j \), \( E_t \) is Young’s modulus of elasticity of the beam material, \( b_i \) is the moment of inertia of the beam cross section, \( \rho \) is the beam mass per unit length, and \( \lambda_i \) is the eigenvalue of mode \( i \).

Equation (9) differs from the conventional uncoupled harmonic oscillator equations of a beam by the terms \( K_{ij}^{\text{PZEO}} \) and \( F_i \). The term \( K_{ij}^{\text{PZEO}} \) represents the passive stiffness contributed by the piezoelectric film. It has the effect of coupling the dynamic modes. The term \( F_i \) is the generalized force associated with mode \( i \). Note that the generalized forces are zero if the electrodes of the piezoelectric film are short circuited. Equation (8) shows the effect of the piezoelectric parameters, the shape function, the mode shape, and the voltage upon the generated modal force.

If the beam were bonded with more than one piezoelectric film, (10) and (11) would be evaluated for each film. The combined passive stiffness terms are determined as follows:

\[ K_{ij}^{\text{PZEO}} = K_{ij}^{\text{PZEO}_1} + K_{ij}^{\text{PZEO}_2} + \cdots + K_{ij}^{\text{PZEO}_m}, \quad \text{for } i = 1, 2, 3, \ldots, n; \quad \text{for } j = 1, 2, 3, \ldots, n \]

(10)

where \( K_{ij}^{\text{PZEO}_k} (k = 1, 2, 3, \ldots, m) \) is obtained by evaluating (10) with the parameters of piezoelectric film PIEZO\(_k\). The combined modal force terms are determined as follows:

\[ F_i^{\Sigma}(t) = F_i^{\text{PZEO}_1}(t) + F_i^{\text{PZEO}_2}(t) + \cdots + F_i^{\text{PZEO}_m}(t), \quad \text{for } i = 1, 2, 3, \ldots, n \]

(11)

where \( F_i^{\text{PZEO}_k}(t) (k = 1, 2, 3, \ldots, m) \) is obtained by evaluating (11) with the parameters of piezoelectric film PIEZO\(_k\) and the voltage \( V^{\text{PZEO}_k} \) across its electrodes. The combined terms are used to formulate the following equation for mode amplitudes:

\[ \ddot{q}_i(t) + \sum_{j=1}^{n} (k_{ij} + K_{ij}^{\Sigma}) \dot{q}_j(t) = F_i^{\Sigma}(t), \quad i = 1, 2, 3, \ldots, n \]

(12)

Substituting (12) into the expression (3) for open-circuit voltage \( V_{oc} \) and (4) for short-circuit current \( I_{sc} \) yields

\[ V_{oc}(t) = \sum_{i=1}^{n} Q_{i}^{V_{oc}} \dot{q}_i(t) \quad \text{and} \quad I_{sc}(t) = \sum_{i=1}^{n} Q_{i}^{I_{sc}} \dot{q}_i(t) \]

where the \( i \)-th modal open circuit voltage \( Q_{i}^{V_{oc}} \) is

\[ Q_{i}^{V_{oc}} := \text{sgn}(\varepsilon_c) \times \frac{d_{33} E_c W_c t_b}{2C_c} \int_{-a}^{a+m} \varepsilon_c(x) \frac{d^2 \Phi_i(x)}{dx^2} dx \]

and the \( i \)-th modal short-circuit current \( Q_{i}^{I_{sc}} \) is

\[ Q_{i}^{I_{sc}} := \text{sgn}(\varepsilon_c) \times \frac{d_{33} E_c W_c t_b}{2} \int_{-a}^{a+m} \varepsilon_c(x) \frac{d^2 \Phi_i(x)}{dx^2} dx. \]

The above equations show the fundamental roles played by the shape function \( \varepsilon_c(x) \) in determining the operation of a strain sensor that measures open-circuit voltage, and a strain rate sensor that measures short-circuit current. The sensor equations have a form that is consistent with the actuator equations.

VII. EXPERIMENTAL RESULTS

Several experiments were performed to verify the interaction equations discussed in Sections V and VI. PVDF films were used to construct three sensor types: a rectangular sensor, a first-mode sensor, and a second-mode sensor. A computer program is used to generate a postscript file that is laser printed to produce a mask for the PVDF film geometry. The PVDF films were bonded to identical aluminum cantilever beams using a Kapton substrate. PZT films were used to construct actuators. PZT films were bonded to aluminum beams using conductive epoxy. Best results were obtained when the PZT was bonded directly to the beam without the use of a Kapton substrate. The experiments are described in detail in [14].

A tip displacement and mid-beam impact test were performed for each shape sensor. During the sensor experiments, the open-circuit voltage and short-circuit currents were measured. The voltages and currents were then compared to MATLAB simulations of both the time response and power spectral densities. In the tip deflection experiment, the beam tip was pushed against an end stop and then released. This gave a repeatable result; however, the vibration is primarily first mode. The mid-beam impact test primarily excites higher modes.

The first-mode and second-mode voltage responses for the tip displacement test are shown in Figs. 7 and 8, respectively. The modal amplitudes corresponding to a 0.02 m tip displacement are \( q_1(0) = 1.367 \times 10^{-2} \) m and \( q_2(0) = -2.730 \times 10^{-6} \) m. This initial condition predominantly excites the first mode, since \( q_1(0) \) is approximately 5000 times larger than the second mode \( q_2(0) \). The modal voltages \( Q_{1}^{V_{oc}} \) and \( Q_{2}^{V_{oc}} \) are a function of a film’s shape. Using a MATLAB program yields \( Q_{1}^{V_{oc}} = -7.95 \times 10^2 \) V/m and \( Q_{2}^{V_{oc}} = -4.71 \times 10^{-2} \) V/m for the first-mode sensor, and \( Q_{1}^{V_{oc}} = -1.50 \times 10^{-2} \) V/m and \( Q_{2}^{V_{oc}} = 1.94 \times 10^4 \) V/m for a second-mode sensor. Ideally, \( Q_{2}^{V_{oc}} = 0 \)
for a first-mode shape film, and \( Q_{1}^{\text{loc}} = 0 \) for a second mode shape film; the discrepancies are due to numerical errors. For the tip test, the expected amplitude is \( Q_{1}^{\text{loc}}q_{1}(0) = 10.9 \) V for the first-mode sensor, and \( Q_{2}^{\text{loc}}q_{2}(0) = 5.3 \times 10^{-2} \) V for the second-mode sensor. This accounts for the attenuation of the first-mode content by the second mode. The first-mode content still leaks through the second-mode sensor due to the difficulty of exactly constructing the required shape.

The actuator response was tested by applying a voltage to a PZT film. An accelerometer was mounted at the tip of the beam. The tip acceleration was compared to the values predicted by simulation with a MATLAB program.

The experiments typically showed that the responses of both the PVDF and the PZT sensors gave voltages and currents that resembled the theoretical values. The discrepancies were due to variations in amplitude. The PVDF voltages were within 93% for the rectangular sensor, 60% for the first mode, and 50% for the second-mode sensor. When used as a sensor, the PZT film gave voltages that were 20% of the expected values; when used as an actuator, the accelerations achieved were also 20% of the expected value. These effects can be attributed to deviations \( d_{33} \), inaccuracies in cutting and etching the PVDF films, and imperfections in bonding both PVDF and PZT films. Fortunately, an empirical calibration of a scale factor for adjusting the actuator and sensor equations can be done using a sensor test.

VIII. CONCLUSIONS

The interaction equations in this paper model the dynamics of flexible substructures with bonded piezoelectric films. The interaction equations allow the required placement and shaping of piezoelectric films to be analyzed separately from the substructure dynamics. The operation of a shaped piezoelectric film is determined by both its placement and its shape function \( \Xi(x) \). The actuator equations have a form that deals with the general case of shaped films, but are substantially simpler than those derived by Crawley and deLuis [5]. The sensor equations have a form that allows signal processing functions to be performed by the appropriate selection of the shape function. The equations have been empirically verified by comparison of computer simulations to the response of a cantilever beam with bonded PVDF and PZT films. An empirical scaling factor can be used to account for variations in \( d_{33} \) and imperfect bonding. Future research will use these interaction equations to evaluate potential control techniques for the active damping of vibrations and shape control of flexible structures.

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REFERENCES


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