International Intertemporal Asset Allocations with Recursive Preference and Mean-Reverting Exchange Rates

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Abstract

Central bank intervention may force the foreign exchange rate to converge to the target zone of the exchange rate, which is periodically reset by the central bank. Therefore, a mean-reversion effect on the exchange rate is observed for foreign exchange markets. An international intertemporal investor with a recursive preference who faces mean-reverting time-varying investment opportunities can consider both myopic demand and intertemporal hedging demand into her asset allocation problem. In this paper we formally evaluate and quantify the mean-reversion effect of exchange rates into an international intertemporal model in order to find the optimal asset allocation strategies and show that the optimal asset allocations can be divided into a myopic component and intertemporal component. We use an asymptotic approximation by taking a first-order expansion around the unit elasticity of intertemporal substitution to derive the exact solution to our optimal problem. Our approximation shows that the intertermporal hedging component first rises then falls with the coefficient of risk aversion. In addition, the magnitude of intertemporal hedging demand increases with a decrease in the mean-reverting intensity on the exchange rate and the elasticity of intertemporal substitution.

Keywords: Mean-reversion in exchange rates, Time-varying opportunities, Intertemporal model

JEL classification: G11, G12

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1. Introduction

As the foreign exchange rate deviates from the target zone of the exchange rate, the central bank may long or short the foreign currency in order to keep the foreign exchange rate stable and steer it into the expected zone of exchange rate of that foreign currency. Central bank intervention may force the exchange rate to be mean-reverting in the target zone of the exchange rate, while at the same time, the central bank may periodically adjust the target zone of the exchange rate since it dynamically changes over time, which results in a time-varying mean-reverting exchange rate environment. Time-varying exchange rates imply that investment opportunities for international investors are time dependent other than time independent. Huizinga (1987), Abuaf and Jorion (1990), Jorion and Sweeney (1996), and Taylor, Peel and Sarno (2001) have found significant evidence that real exchange rates are mean-reverting, especially at long horizons. Due to deregulation, investors face a globally diversifiable investment universe where domestic and foreign securities are both considered into their international intertemporal asset allocation problem. Merton (1971, 1973) assumed that asset returns are stochastic with constant parameters when only considering the domestic market, however, the importance of long-horizon predictability in an international context has been stressed earlier by Culter, Poterba, and Summers (1989). There is now considerable evidence that excess returns in the international equity markets or in the foreign exchange markets are predictable. International intertemporal asset pricing models with constant exchange rate parameters may lead investors to an incorrect allocation of investment, considering that a mean-reverting phenomenon of exchange rates may overcome the constant parameter problems in the dynamic investment environments.

In a log preference shown by Zapatero (1995), where investors act myopically, the intertemporal model is tractable and reduced to the single-period model. Since the elasticity of intertemporal substitution and the relative risk aversion have very different aspects on optimal consumption and portfolio choice (Campbell and Viceira, 1999, Campbell, Chacko, Rodriguez and Viceira, 2004, and Chacko and Viceira, 2005), a continuous-time recursive preference shown by

Duffie and Epstein (1992b) can be used to distinguish the relative risk aversion from the elasticity of intertemporal substitution in consumption. Thus, this paper develops a continuous-time recursive preference international intertemporal model in order to clarify both the risk aversion effect and intertemporal substitution effect on the optimal asset allocation decisions.

Facing a mean-reverting environment of an exchange rate, two long-lived representative agents with recursive preference are considered in both the domestic and foreign country. Each representative agent of the domestic country and foreign country maximizes her utility which is defined over consumption rather than wealth, and chooses an optimal allocation on international securities as well as consumption from her wealth. However, an exact closed-form solution does not appear possible under the endogeneity of consumption in our model.

Several studies solve the problem by some approximation methods. Liu (2001) set up a square root short rate diffusion process and stochastic volatility on stock returns to show that Merton's optimal dynamic portfolio selection problem expressed by a non-linear partial differential equation can be reduced to the system of ordinary differential equations. Campbell and Viceira (2001) use log linear approximation to characterize the portfolio demand under a stochastic opportunity set in a discrete time model; Campbell, Chacko, Rodriguez and Viceira (2004) and Chacko and Viceira (2005) derive the optimal portfolio with stochastic opportunity sets by an approximate analytical solution method and find a solution around a particular point in the state space—the unconditional mean of the log consumption-wealth ratio. In contrast to them, we derive the exact solution by a more intuitive perturbation method of approximation around a particular point in the preference space.

In order to make the analysis herein more tractable, financial markets are assumed to be fully integrated and there are no constraints on international capital flows. This goes along with a certain number of general assumptions of market perfection, including: capital markets are always in equilibrium; assets can be sold short; there exists a bond market for borrowing and lending at the same rate; and trading in assets and currencies takes place continuously in time. Some aspects, such as human capital and inflation, are important for international investment, but are left out since they might blur the focus of this paper.

Our model formally evaluates and quantifies the recursive preference into an international intertemporal asset allocation problem under mean-reverting exchange rates. More specifically, our contributions consist of the following. First, some recent studies all present their dynamic portfolios under the time-varying opportunities sets. Bekaert and Hodrick (1992) investigate the equity and foreign exchange excess returns with a vector autoregressive process. Campbell and Viceira (1999) model the time-varying equity premium using a discrete-time model and present an approximate analytical solution to analyze the impact of a predictable variation in the stock return on an intertemporal optimal portfolio choice and consumption. Campbell, Chacko, Rodriguez and Viceira (2004) find a continuous-time representation of Campbell and Viceira (1999) and exhibit the same properties as its discrete-time counterpart. All the papers mentioned above, together with Brennan, Schwartz and Lagnado (1997), Liu (2001), and Wachter (2002) are limited in national investment decisions other than international asset allocations, and therefore we extend our model from a national to an international setting.

Second, although Zapatero (1995) already considers international intertemporal asset allocation problems, the log utility setting has induced Zapatero (1995)'s model to reduce to a single period international asset allocation model. Zapatero's model explains the current myopic investment of investors only, while leaving unexplained the intertemporal hedging demand of investors. By observing the limitation of Zapatero, we present a continuous-time recursive preference model which allows us not only to analyze the effect of risk aversion on investors' asset allocation decision, but also to realize the intertemporal substitution effect in consumption. Our model accounts for myopic hedging as well as intertemporal hedging strategies in the spirit of the model tested by Dumas and Solnik (1992) and Harvey, Solnik and Zhou (1992). Furthermore, our model allows for differences in beliefs across the representative agents expressed by the time-varying investment opportunities sets. This generalization allows us to analyze the comparative analysis on the parameters of the optimal allocations.

Third, under the endogeneity of consumption in an intertemporal asset allocation model, an explicit closed-form solution in the general case does not appear possible. In the situation where an investor considers intermediate consumption, the approximation method provided by Campbell and Viceira (2001), Campbell, Chacko, Rodriguez, and Viceira (2004), and Chacko and Viceira (2005) - through the approximation around the unconditional mean of the log consumption-wealth ratio - is accurate only in the situation whereby the log consumption-wealth ratio is not too variable around its unconditional mean. Therefore, we derive the explicit solution by a more intuitive perturbation method of approximation around a particular point in the preference space. Here, the intertemporal elasticity equal to 1.

Fourth, we find some numerical results which are shown as follows. (1) The magnitude of intertemporal hedging demand first rises and then falls with an increase in the coefficient of risk aversion. Accordingly, when the investor becomes extremely conservative, her optimal intertemporal hedging demand on the foreign risky stock expressed in domestic currency will decrease to zero. (2) On determining the optimal weight in the intertemporal hedging demand, the relative risk aversion is more sensitive than the elasticity of intertemporal substitution, while the volatility of intertemporal hedging demand shows a flattening trend with an increase in the elasticity of intertemporal substitution. (3) The magnitude of intertemporal hedging demand decreases under the increasing intensity of mean reversion, however, the mean reversion sensitivity on intertemporal hedging demand increases as risk aversion decreases.

The paper is organized as follows. Section 2 describes the international financial economy and the model. Section 3 develops the optimal dynamic asset allocation strategies for long-horizon investors in the time-varying international environment. Section 4 presents a numerical exercise on general international intertemporal asset allocation problems. Finally, section 5 presents some conclusions.

2. An international financial economy

We consider a financial economy consisting of two countries, D and F (representing domestic and foreign), each populated by a representative agent. Financial markets are assumed to be fully integrated, and trades take place between investors through a foreign exchange rate following the setting of Solnik (1974), Stulz (1981), Adler and Dumas (1983), and Zapatero (1995).

2.1 Rates of return dynamics and exchange rate structure

An investor in each country faces two domestic assets, a riskless bond and a risky stock, and two foreign assets. It is assumed that expectations in real terms are homogeneous across both investors (Solnik, 1974). Letting S_t^D denote the price of the domestic risky asset at time t, the return dynamics are given by:

$$\begin{aligned}\frac{dS_t^D}{S_t^D} &= \mu_D dt + s_{DI} dZ_I + s_{DD} dZ_D \\ &\equiv \mu_D dt + \boldsymbol{\sigma}'_{\mathbf{D}} \mathbf{dZ},\end{aligned} \tag{1}$$

where $\frac{dS_t^D}{S_t^D}$ is the instantaneous rate of return on the risky asset of country D; μ_D is the instantaneous expected rate of return on the stock; $\boldsymbol{\sigma}'_{\mathbf{D}} \equiv [s_{DI}\ s_{DD}\ 0]$ represents the diffusion where s_{DI} and s_{DD} are volatilities respectively due to the international factor and the idiosyncratic factor in home country D; and $\mathbf{dZ}' \equiv [dZ_I\ dZ_D\ dZ_F]$ is a vector of Wiener processes. The second asset that the representative agent faces is the riskless bond. Here, B_t denotes the price of the bond at time t, and its stochastic process is given by:

$$\frac{dB_t^D}{B_t^D} = r^D dt\ , \tag{2}$$

where r^D is the interest rate in country D.

The financial market of country F are similar to Equations (1) and (2), with parameters μ_F, s_{FI}, s_{FF}, $\sigma'_F \equiv [s_{FI}\ 0\ s_{FF}]$, and r^F. The dynamics of the risky technology depend only on Z_I and Z_F. We assume, following the setting on Zapatero (1995), that there are a total of three factors to explain the dynamics of the financial economy. One is common across countries (denoted by Z_I) and the other two are idiosyncratic to each country (denoted by Z_D and Z_F for countries D and F, respectively). Therefore, domestic financial markets are incomplete in each country.

Trade across countries takes place through an exchange rate. As we know, central banks in most countries may trade foreign currencies frequently in order to stabilize foreign exchange rates. When the foreign exchange rate exceeds the ceiling or is below the floor of the target exchange rate zone, a central bank may short or long the foreign currency in order to push the foreign exchange rate back into the expected target zone of that foreign currency. Central bank intervention causes a mean-reverting phenomenon on foreign exchange rates, and when facing a dynamic environment, the central bank may periodically reset the target zone of the exchange rates which will induce the foreign exchange rates to be time-varying mean-reverted. Therefore, investment opportunities faced by international investors are in fact time dependent other than time independent in a time-varying exchange rate environment. Indeed, Huizinga (1987), Abuaf and Jorion (1990), Jorion and Sweeney (1996), and Taylor, Peel and Sarno (2001) find that real exchange rates are significantly mean-reverting in the long term, especially for long horizons.

The percentage rate of changes in the exchange rate is assumed to be driven by three independent Brownian processes, Z_I, Z_D, and Z_F, with a mean-reverting drift. The dynamics of the rate of exchange rate are given as follows:

$$\begin{aligned}\frac{de_t}{e_t} &= \mu_{et}dt + s_{eI}dZ_I + s_{eD}dZ_D + s_{eF}dZ_F \\ &\equiv \mu_{et}dt + \boldsymbol{\sigma}'_{\mathbf{e}}\mathbf{dZ_e}, \end{aligned} \tag{3}$$

where $d\mu_{et} = \kappa(\theta - \mu_{et})dt + \sigma_{\mu_e} d\widetilde{Z}_{\mu_e}$. Here, $\theta \in (\theta_L, \theta_H)$ is the long-term mean of the expected changes in the exchange rate, which can be interpreted as the target zone of the exchange rate. The target zone of the exchange rate is set by the central bank in order to maintain the foreign currency value within the central bank's exchange rate policy. When $e_t > \theta_H$, the current exchange rate exceeds the upper bound of the target zone, and foreign currency will have to depreciate in value, which results in a decline in next exchange rate. On the other hand, if $e_t < \theta_L$, then the foreign currency will have to appreciate in value.

While κ is the mean reversion intensity of the expected rate of changes in the exchange rate, a higher κ indicates a higher reversion speed of the next exchange rate move toward the target zone. Therefore, the foreign currency market is more volatile than ever, and the persistence of the foreign exchange rate is thus lower. By observing the mean-reverting phenomenon in foreign exchange rates, if $e_t > \theta_H$, then e_{t+1} will decrease, and κ should be positive in order to realize a negative payoff in the next period. If $e_t < \theta_L$, then e_{t+1} will increase, and κ should be positive in order to realize a positive payoff in the next period. Therefore, κ is positive no matter if the foreign exchange rate is higher or lower than the target zone.

The shocks to the expected rate of change in the exchange rate $d\mu_{et}$ and $\frac{de_t}{e_t}$ are given by

$d\tilde{Z}_{\mu_e} = \rho_I dZ_I + \rho_D dZ_D + \rho_F dZ_F + (\sqrt{1-\rho_I^2-\rho_D^2-\rho_F^2})dZ_{\mu_e} \equiv \boldsymbol{\rho}'\mathbf{dZ} + \left(\sqrt{1-\boldsymbol{\rho}'\boldsymbol{\rho}}\right)dZ_{\mu_e}$ and $\mathbf{dZ_e} = \mathbf{dZ}$, respectively. The three independent Brownian motion processes which derive the exchange rate can be used to explain all the uncertainty in the financial economy. In addition, the expected rate of change in the exchange rate varies over time with a mean-reverting process. Note that the instantaneous correlation vector between de_t/e_t and $d\mu_{et}$ is $\boldsymbol{\rho} \equiv [\rho_I \;\; \rho_D \;\; \rho_F]'$, and ρ_I, ρ_D, and ρ_F are instantaneous correlations between the time-varying expected rate of change in the exchange rate and volatilities due to the common international factor, and the idiosyncratic factors in country D and country F, respectively.

2.2 Preference structure and budget equations dynamics

In order to capture the intertemporal hedging demand on the asset allocation problem, the recursive utility, described by Campbell (1993), Campbell and Viceira (1999, 2002), Campbell, Chacko, Rodriguez and Viceira (2004), and Chacko and Viceira (2005), is used to describe investors' preferences. Epstein and Zin (1989, 1991) first derive a parameterization of recursive utility in a discrete-time setting, while Duffie and Epstein (1992a, 1992b) offer a continuous-time setting. The recursive utility function allows us to separates the relative risk aversion coefficient from the elasticity of the intertemporal substitution in consumption, which means it separates the investor's risk attitude on optimal consumption and the portfolio from the consuming attitude on the single consumption good. For our representative agents in countries D and F, the recursive preference over consumption is given by:

$$J^i = E_t[\int_t^\infty f(C_\tau^i, J_\tau^i)\, d\tau] \quad i \in [D, F], \tag{4}$$

where $f(C_\tau^i, J_\tau^i)$ is a normalized aggregator of investors' current consumption (C_τ^i) and utility that take the following form:

$$f(C^i, J^i) = \beta^i (1 - \frac{1}{\varphi^i})^{-1} (1-\gamma^i) J^i \left[\left(\frac{C^i}{((1-\gamma^i)J^i)^{\frac{1}{1-\gamma^i}}} \right)^{1-\frac{1}{\varphi^i}} - 1 \right] \; i \in [D, F]. \tag{5}$$

Here, $\gamma^i > 0$ is the coefficient of relative risk aversion, $\beta^i > 0$ is the rate of time preference, and $\varphi^i > 0$ is the elasticity of intertemporal substitution. They are all larger than zero. When the elasticity of the intertemporal substitution is the reciprocal of risk aversion $\varphi^i = (\gamma^i)^{-1}$, the recursive utility function reduces to the standard, additive power utility function. Therefore, the power utility function is just a special case of the recursive utility function.

The exchange rate dynamics together with the asset return dynamics yield a return process of a foreign asset expressed in domestic currency. Since the return of a foreign asset expressed in domestic currency depend not only on the change in the price of the foreign asset, but also on the change in the exchange rate, the return of a foreign asset expressed in domestic currency is therefore different from the return of that foreign asset expressed in foreign currency. When investors have access to the international market, foreign assets become available to them, and the actual returns the international investors face are different from those of the domestic investors, because the exchange rate changes over time.

By Ito's Lemma, the instantaneous return for the representative investor of country D in an investment on a stock and bond in country F is respectively:

$$\frac{d(e_t S_t^F)}{e_t S_t^F} = (\mu_F + \mu_{et} + \boldsymbol{\sigma}'_{\mathbf{F}}\boldsymbol{\sigma}_{\mathbf{e}})dt + (\boldsymbol{\sigma}'_{\mathbf{F}} + \boldsymbol{\sigma}'_{\mathbf{e}})\mathbf{dZ} \tag{6}$$

and

$$\frac{d(e_t B_t^F)}{e_t B_t^F} = (r^F + \mu_{et})dt + \boldsymbol{\sigma}'_{\mathbf{e}}\mathbf{dZ}\,. \tag{7}$$

Similarly, for the representative agent of country F, the instantaneous return on investing in a risky stock and local riskless bond of country D is respectively:

$$\frac{d(S_t^D / e_t)}{S_t^D / e_t} = (\mu_D - \mu_{et} + \boldsymbol{\sigma}'_{\mathbf{e}}\boldsymbol{\sigma}_{\mathbf{e}} - \boldsymbol{\sigma}'_{\mathbf{D}}\boldsymbol{\sigma}_{\mathbf{e}})dt + (\boldsymbol{\sigma}'_{\mathbf{D}} - \boldsymbol{\sigma}'_{\mathbf{e}})\mathbf{dZ}\,, \tag{8}$$

and

$$\frac{d(B_t^D / e_t)}{B_t^D / e_t} = (r^D - \mu_{et} + \boldsymbol{\sigma}'_{\mathbf{e}}\boldsymbol{\sigma}_{\mathbf{e}})dt - \boldsymbol{\sigma}'_{\mathbf{e}}\mathbf{dZ}\,. \tag{9}$$

The investor's objective is to maximize the expected lifetime utility described above, subject to the following intertemporal budget constraint. The intertemporal budget constraint for the representative agent in country D is:

$$\begin{aligned} dW_t^D &= [n_t^{DD}(\mu_D - r^D) + n_t^{DF}(\mu_F + \mu_{et} + \boldsymbol{\sigma}'_{\mathbf{F}}\boldsymbol{\sigma}_{\mathbf{e}} - r^D) \\ &\quad + n_t^{DB}(\mu_{et} + r^F - r^D)]W_t^D dt + [n_t^{DD}\boldsymbol{\sigma}'_{\mathbf{D}} + n_t^{DF}(\boldsymbol{\sigma}_{\mathbf{F}} + \boldsymbol{\sigma}_{\mathbf{e}})' + n_t^{DB}\boldsymbol{\sigma}'_{\mathbf{e}}]\mathbf{dZ}W_t^D \\ &\quad + r^D W_t^D dt - C_t^D dt\,. \end{aligned} \tag{10a}$$

For the representative agent in country F, the intertemporal budget constraint is:

$$\begin{aligned} dW_t^F &= [n_t^{FD}(\mu_D - \mu_{et} + \boldsymbol{\sigma}'_{\mathbf{e}}\boldsymbol{\sigma}_{\mathbf{e}} - \boldsymbol{\sigma}'_{\mathbf{D}}\boldsymbol{\sigma}_{\mathbf{e}} - r^F) + n_t^{FF}(\mu_F - r^F) \\ &\quad + n_t^{FB}(\boldsymbol{\sigma}'_{\mathbf{e}}\boldsymbol{\sigma}_{\mathbf{e}} - \mu_{et} + r^D - r^F)]W_t^F dt + [n_t^{FD}(\boldsymbol{\sigma}_{\mathbf{D}} - \boldsymbol{\sigma}_{\mathbf{e}})' + n_t^{FF}\boldsymbol{\sigma}'_{\mathbf{F}} - n_t^{FB}\boldsymbol{\sigma}'_{\mathbf{e}}]\mathbf{dZ}W_t^F \\ &\quad + r^F W_t^F dt - C_t^F dt\,, \end{aligned} \tag{10b}$$

where W_t^i, $i \in [D, F]$ represents the investor's total wealth in countries D and F. Following Solnik (1974), investors of both countries are assumed to consume a single domestic consumption good in each time t. Moreover, n_t^{iD}, n_t^{iF}, and n_t^{iB}, where $i \in [D, F]$, are the fractions of investors' wealth allocated to financial assets: the stock of country D, the stock of country F, and the local riskless bond of the foreign country, respectively.

3. The optimal dynamic asset allocation strategies

3.1 The optimality and the exact portfolio choice for the special case: intertemporal substitution of consumption with unit elasticity

Investors maximize the expected lifetime utility expressed by the value functions of the problem (J^i). In fact, the principle of optimality leads to the following Bellman equations of the utility function for the representative investors of country D and country F which satisfy:

$$0=\sup_{\{n_t^{i'},C_t^i\}}\left\{f(C_\tau^i,J_\tau^i)+J_{W^i}^i\left[\mathbf{n_t^i}'(\mathbf{R^i}-r^i\mathbf{1})W_t^i+r^iW_t^i-C_t^i\right]+J_{\mu_{et}}^i\kappa(\theta-\mu_{et})\right.$$
$$\left.+\frac{1}{2}J_{W^iW^i}^i(\mathbf{n_t^i}'\mathbf{V^i}'\mathbf{V^i}'\mathbf{n_t^i}')(W_t^i)^2+J_{W^i\mu_{et}}^i(\mathbf{n_t^i}'\mathbf{V^i}\,\boldsymbol{\rho}\sigma_{\mu_e})W_t^i+\frac{1}{2}J_{\mu_{et}\mu_{et}}^i(\sigma_{\mu_e})^2\right\}\qquad i\in[D,F],\quad (11)$$

whereby we denote:

$\mathbf{n_t^D}'\equiv[n_t^{DD}n_t^{DF}n_t^{DB}]$,

$\mathbf{n_t^F}'\equiv[n_t^{FD}n_t^{FF}n_t^{FB}]$,

$\mathbf{V^D}\equiv[\boldsymbol{\sigma}'_\mathbf{D}\quad(\boldsymbol{\sigma}_\mathbf{F}+\boldsymbol{\sigma}_\mathbf{e})'\quad\boldsymbol{\sigma}'_\mathbf{e}]'$,

$\mathbf{V^F}\equiv[(\boldsymbol{\sigma}_\mathbf{D}-\boldsymbol{\sigma}_\mathbf{e})'\quad\boldsymbol{\sigma}'_\mathbf{F}\quad(-\boldsymbol{\sigma}'_\mathbf{e})]'$,

$(\mathbf{R^D}-r^D\mathbf{1})\equiv\left[(\mu_D-r^D)\quad(\mu_F+\mu_{et}+\boldsymbol{\sigma}'_\mathbf{F}\boldsymbol{\sigma}_\mathbf{e}-r^D)\quad(\mu_{et}+r^F-r^D)\right]'$,

$(\mathbf{R^F}-r^F\mathbf{1})\equiv\left[(\mu_D-\mu_{et}+\boldsymbol{\sigma}'_\mathbf{e}\boldsymbol{\sigma}_\mathbf{e}-\boldsymbol{\sigma}'_\mathbf{D}\boldsymbol{\sigma}_\mathbf{e}-r^F)\quad(\mu_F-r^F)\quad(\boldsymbol{\sigma}'_\mathbf{e}\boldsymbol{\sigma}_\mathbf{e}-\mu_{et}+r^D-r^F)\right]'$,

$J_{W^i}^i$ and $J_{\mu_{et}}^i$ denote the derivatives of J^i with respect to W^i and μ_{et}, and similar notations are used for higher derivatives.

In order to derive the optimal weights on the consumption good and financial assets, the first-order conditions on the above Bellman equations are presented as:

$$C_t^i=(J_{W^i}^i)^{-1}J^i\beta^i(1-\gamma^i),\qquad (12)$$

$$\mathbf{n_t^i}=\frac{-J_{W^i}^i}{J_{W^iW^i}^iW_t^i}(\mathbf{V^i}\,\mathbf{V^i}')^{-1}(\mathbf{R^i}-r^i\mathbf{1})+\frac{-J_{W^i\mu_{et}}^i}{J_{W^iW^i}^iW_t^i}(\mathbf{V^i}\,\mathbf{V^i}')^{-1}\mathbf{V^i}\boldsymbol{\rho}\,\sigma_{\mu_e}\qquad i\in[D,F].\qquad (13)$$

Currently, the first-order conditions for our problem are not explicit solutions unless we know the complicated form of the indirect utility function. We conjecture a solution of the indirect utility function as:

$$J^i(W_t^i,\mu_{et}) = I^i(\mu_{et})\frac{(W_t^i)^{1-\gamma^i}}{1-\gamma^i}. \tag{14}$$

However, the explicit solutions on the first-order conditions still cannot be found unless we know the solution of the direct utility function. Therefore, under unit elasticity of an intertemporal substitution of consumption $\varphi^i = 1$, we guess that the ordinary differential equation has a solution of the functional form:

$$I^i(\mu_{et}) = \exp[\hat{Q}_0^i + \hat{Q}_1^i\mu_{et} + \frac{1}{2}\hat{Q}_2^i(\mu_{et})^2], \quad i \in [D,F]. \tag{15}$$

Using our conjectures on the functional form of the indirect utility function (Equations (14) and (15)), we then substitute the first-order conditions (Equations (12) and (13)) back into the Bellman equation (Equation (11)) and rearrange to get:

$$\begin{aligned}
0 &= \left(\log\beta^i - \frac{1}{1-\gamma^i}[\hat{Q}_0^i + \hat{Q}_1^i\mu_{et} + \frac{1}{2}\hat{Q}_2^i(\mu_{et})^2] - 1\right)\beta^i + r^i \\
&+ (\hat{Q}_1^i + \hat{Q}_2^i\mu_{et})\frac{1}{1-\gamma^i}\kappa(\theta - \mu_{et}) + \frac{1}{2}\frac{1}{\gamma^i}(\mathrm{R}^{\mathrm{i}} - r^i 1)'(\mathrm{V}^i\,\mathrm{V}^{i'})^{-1}(\mathrm{R}^{\mathrm{i}} - r^i 1) \\
&+ \frac{1}{\gamma^i}(\hat{Q}_1^i + \hat{Q}_2^i\mu_{et})(\mathrm{R}^{\mathrm{i}} - r^i 1)'(\mathrm{V}^i\,\mathrm{V}^{i'})^{-1}\mathrm{V}^i\rho\,\sigma_{\mu_e} \\
&+ \frac{1}{2}\frac{1}{\gamma^i}[(\hat{Q}_1^i)^2 + 2\hat{Q}_1^i\hat{Q}_2^i\mu_{et} + (\hat{Q}_2^i)^2(\mu_{et})^2]\sigma_{\mu_e}^2\,\rho'\,\mathrm{V}^{i'}(\mathrm{V}^i\,\mathrm{V}^{i'})^{-1}\,\mathrm{V}^i\rho \\
&+ \frac{1}{2}\frac{1}{1-\gamma^i}[(\hat{Q}_1^i)^2 + 2\hat{Q}_1^i\hat{Q}_2^i\mu_{et} + (\hat{Q}_2^i)^2(\mu_{et})^2 + \hat{Q}_2^i]\sigma_{\mu_e}^2 \qquad i \in [D,F]
\end{aligned} \tag{16}$$

Rearranging the above equation, the system of the three recursive equations of $\hat{Q}_2^D$, $\hat{Q}_1^D$ and $\hat{Q}_0^D$ results from collecting terms in $(\mu_{et})^2$, μ_{et}, and the constant terms for the representative agent in country D. Similarly, we have another system of three recursive equations of $\hat{Q}_2^F$, $\hat{Q}_1^F$ and $\hat{Q}_0^F$, which also result from the collecting terms in $(\mu_{et})^2$, μ_{et}, and the constants for the representative agent in country F. All the recursive equations of $\hat{Q}_2^i$, $\hat{Q}_1^i$ and $\hat{Q}_0^i$, $i \in [D,F]$, for the representative agent in countries D and F are summarized in Appendix A.

When the elasticity of intertemporal substitution is restricted to one $\varphi^i = 1$, an investor's optimal instantaneous consumption and optimal dynamic asset allocation strategies can be derived as:

$$\frac{C_t^i}{W_t^i} = \beta^i, \qquad i \in [D, F], \tag{17}$$

and

$$\mathbf{n_t^i} = \frac{1}{\gamma^i}(\mathbf{V}^i\, \mathbf{V}^{i\prime})^{-1}(\mathbf{R}^{\mathrm{i}} - r^i\mathbf{1}) + (1 - \frac{1}{\gamma^i})\frac{\hat{Q}_1^i + \hat{Q}_2^i \mu_{et}}{\gamma^i - 1}(\mathbf{V}^i\, \mathbf{V}^{i\prime})^{-1}\mathbf{V}^i \boldsymbol{\rho}\, \sigma_{\mu_e} \qquad i \in [D, F]. \tag{18}$$

In the special case of $\varphi^i = 1$, the investor's optimal consumption-wealth ratio equals the discount factor. The optimal consumption-wealth ratio in our financial economy is not state dependent, which means our optimal consumption weight depends only on the investor's time preference, not on the state variable μ_{et}. Therefore, the investor's risk attitude and the intertemporal hedging considerations are not the determinant of the optimal consumption. Additionally, after allocating her wealth on current consumption, Equation (18) shows us that investors in countries D and F respectively allocate $\frac{1}{\gamma^i}$ and $1 - \frac{1}{\gamma^i}, i \in [D, F]$ stakes of their wealth on the financial assets in order to satisfy their myopic demand and intertemporal hedging demand in the case of $\varphi^i = 1$.

The myopic demand is for an investor who invests only in a single period horizon or faces a constant investment opportunity set. In our optimal asset allocation strategies, the myopic demand $\frac{1}{\gamma^i}(\mathbf{V}^i\, \mathbf{V}^{i\prime})^{-1}(\mathbf{R}^{\mathrm{i}} - r^i\mathbf{1})$ is determined by the mean-variance efficiency of the opportunity set and is expressed as the inverse of the variance-covariance matrix of the world market portfolio in a given currency. Since the expected changes of exchange rates are time-varying, the mean-variance matrix on our optimization problem is time-varying, and therefore the myopic component is also time-varying. On the other hand, intertemporal hedging demand characterizes the demand arising from the desire to hedge against changes in the time-varying investment opportunity sets. The intertemporal hedging components of the optimal asset allocation $(1-\frac{1}{\gamma^i})\frac{\hat{Q}_1^i+\hat{Q}_2^i\mu_{et}}{\gamma^i-1}(\mathbf{V}^i\,\mathbf{V}^{i'})^{-1}\mathbf{V}^i\boldsymbol{\rho}\,\sigma_{\mu_e}$ are affine functions of the time-varying expected rates of the changes in exchange rates with coefficients $\hat{Q}_1^i$ and $\hat{Q}_2^i$, $i\in[D,F]$. The intertemporal hedging demand can be determined by the product of the coefficient of risk aversion γ^i, the variance-covariance matrix $(\mathbf{V}^i\,\mathbf{V}^{i'})$, the covariance between risky assets and states $\mathbf{V}^i\boldsymbol{\rho}\,\sigma_{\mu_e}$, as well as the instantaneous rates of changes of the value function $\hat{Q}_1^i+\hat{Q}_2^i\mu_{et}$.

The impact of pure changes in the time-varying exchange rates on intertemporal hedging components depends on the signs of $\hat{Q}_1^i$ and $\hat{Q}_2^i$. Since $\hat{Q}_1^i$ is recursively determined by $\hat{Q}_2^i$, we first determine the sign of $\hat{Q}_2^i$. In the case of $\gamma=\varphi=1$, Campbell and Viceira (1999, 2002), Campbell, Chacko, Rodriguez and Viceira (2004), and Chacko and Viceira (2005) show that the only root that maximizes the value function is $\hat{Q}_2^i=0$, which means when $\gamma=\varphi=1$, the recursive preference reduces to be the log preference. Moreover, in the more general case where $\gamma^i>1$, as shown in Appendix A Equation (A5), coefficient $\hat{Q}_2^i$ has two real roots of opposite signs according to the quadratic equation theory. In each quadratic equation, we would like to know which solution is good for our problem from the criteria that the roots of the discriminant are real. The value function J is maximized only with the solution associated with the negative root of Equation (A5) - that is, the value function J - is maximized only with the solution that $\hat{Q}_2^i<0$ when $\gamma^i>1$. Moreover, the intercept is negative $\hat{Q}_1^i<0$ when $\hat{Q}_2^i<0$ and is independent on the level of the time-varying expected rates of changes in the exchange rates.

Since $\hat{Q}_1^i<0$, $\hat{Q}_2^i<0$ when $\gamma^i>1$, the sign of the intertemporal hedging demand depends only on the sign of the covariance between risky assets and states. When $\mathbf{V}^i\boldsymbol{\rho}\,\sigma_{\mu_e}>0$, the intertemporal hedging demand is negative, which means that risky assets yield no hedging ability, and therefore investors decrease their total allocation on financial assets. On the contrary, when $\mathbf{V}^i \boldsymbol{\rho}\, \sigma_{\mu_e} < 0$, the intertemporal hedging demand on optimal asset allocation is positive, and investors will allocate more on the intertemporal stake since risky assets can be used as hedging tools.

3.2 The generalized solutions of the optimalization $\varphi^i \neq 1$

When the elasticity of intertemporal substation is not restricted to one, the first-order condition for consumption is different from Equation (12) and it will be:

$$C_t^i = (J_{W^i}^i)^{-\varphi^i} (J^i)^{\frac{1-\varphi^i \gamma^i}{1-\gamma^i}} (\beta^i)^{\varphi^i} (1-\gamma^i)^{\frac{1-\varphi^i \gamma^i}{1-\gamma^i}}. \tag{19}$$

The first-order condition for financial assets is the same as Equation (13), and as we substitute the first-order conditions of Equations (19) and (13) back into the Bellman equation, the exact solution to our optimal problem does not appear possible. Campbell (1993), Judd and Guu (1997), Kogan and Uppal (2001), Campbell and Viceira (1999, 2002), Campbell, Chacko, Rodriguez and Viceira (2004), and Chacko and Viceira (2005) use perturbation approaches to solve the dynamic financial optimization problem. Our paper also adopts a more intuitive perturbation method to solve the generalized optimization problem - that is, the first-order expansion of the Bellman equation we implement is around the elasticity of intertemporal substitution. The approximation is written as:

$$(I^i)^{1+\frac{1-\varphi^i}{1-\gamma^i}} \approx I^i + (I^i)^{1+\frac{1-\varphi^i}{1-\gamma^i}} \log\left(I^i \cdot \frac{(-1)}{1-\gamma^i} \right)\Bigg|_{\varphi^i=1} \times (\varphi^i - 1) = I^i + \frac{1-\varphi^i}{1-\gamma^i} I^i \log I^i \quad i \in [D,F]. \tag{20}$$

Substituting Equation (20) back into the Bellman equation and guessing the ordinary differential equation present a solution of the form:

$$I^i(\mu_{et}) = \exp[\widetilde{Q}_0^i + \widetilde{Q}_1^i \mu_{et} + \frac{1}{2} \widetilde{Q}_2^i (\mu_{et})^2], \qquad i \in [D,F]. \tag{21}$$

The approximating Bellman equation can be expressed as:

$$0=-\frac{(\beta^i)^{\varphi^i}}{1-\varphi^i}-\frac{(\beta^i)^{\varphi^i}}{1-\gamma^i}[\widetilde{Q}_0^i+\widetilde{Q}_1^i\mu_{et}+\frac{1}{2}\widetilde{Q}_2^i(\mu_{et})^2]+\frac{\varphi^i}{1-\varphi^i}\beta^i+r^i+(\widetilde{Q}_1^i+\widetilde{Q}_2^i\mu_{et})\frac{1}{1-\gamma^i}\kappa(\theta-\mu_{et})$$
$$+\frac{1}{2}\frac{1}{\gamma^i}(\mathrm{R}^i-r^i1)'(\mathrm{V}^i\,\mathrm{V}^{i'})^{-1}(\mathrm{R}^i-r^i1)+\frac{1}{\gamma^i}(\widetilde{Q}_1^i+\widetilde{Q}_2^i\mu_{et})(\mathrm{R}^i-r^i1)'(\mathrm{V}^i\,\mathrm{V}^{i'})^{-1}\mathrm{V}^i\rho\,\sigma_{\mu_e}$$
$$+\frac{1}{2}\frac{1}{\gamma^i}[(\widetilde{Q}_1^i)^2+2\widetilde{Q}_1^i\widetilde{Q}_2^i\mu_{et}+(\widetilde{Q}_2^i)^2(\mu_{et})^2]\sigma_{\mu_e}^2\,\rho'\,\mathrm{V}^{i'}(\mathrm{V}^i\,\mathrm{V}^{i'})^{-1}\,\mathrm{V}^i\rho \tag{22}$$
$$+\frac{1}{2}\frac{1}{1-\gamma^i}[(\widetilde{Q}_1^i)^2+2\widetilde{Q}_1^i\widetilde{Q}_2^i\mu_{et}+(\widetilde{Q}_2^i)^2(\mu_{et})^2+\widetilde{Q}_2^i]\sigma_{\mu_e}^2 \qquad i\in[D,F]$$

Rearranging the above equation, as shown in Appendix B Equation (B1) to (B3), we have three recursive equations of $\widetilde{Q}_2^D$, $\widetilde{Q}_1^D$ and $\widetilde{Q}_0^D$ that result from the collecting terms in $(\mu_{et})^2$, μ_{et}, and the constants for the representative agent in country D. After this, we can really get the indirect utility function and the investors' optimal consumption policies and their dynamic asset allocation strategies in the time-varying international investment environment.

When $\varphi^i \neq 1$, by taking the first-order expansion around the unit elasticity of intertemporal substitution, the investor's optimal instantaneous consumption-wealth ratio is expressed as:

$$\frac{C_t^i}{W_t^i}=(\beta^i)^{\varphi^i}\exp\{(\frac{1-\varphi^i}{1-\gamma^i})[\widetilde{Q}_0^i+\widetilde{Q}_1^i\mu_{et}+\frac{1}{2}\widetilde{Q}_2^i(\mu_{et})^2]\}, \tag{23}$$

while the investor's optimal dynamic asset allocation strategies are:

$$\mathbf{n}_t^i=\frac{1}{\gamma^i}(\mathbf{V}^i\,\mathbf{V}^{i'})^{-1}(\mathbf{R}^i-r^i\mathbf{1})+(1-\frac{1}{\gamma^i})(\frac{\widetilde{Q}_1^i+\widetilde{Q}_2^i\,\mu_{et}}{(\gamma^i-1)})(\mathbf{V}^i\,\mathbf{V}^{i'})^{-1}\mathbf{V}^i\boldsymbol{\rho}\,\sigma_{\mu_e} \qquad i\in[D,F]. \tag{24}$$

If the elasticity of intertemporal substitution is not restricted to one, then the optimal consumption and optimal weight on financial assets are all state dependent. First, the time preference β, the risk attitude of investor γ, the intertemporal substitution φ, and the state variable μ_{et} are the determinants of the optimal consumption, and therefore in the general case, the optimal consumption weight is state dependent and time-varying. Second, investors in countries D and F respectively spend $\frac{1}{\gamma^i}$ and $1-\frac{1}{\gamma^i}, i\in[D,F]$ stakes of their wealth on the financial assets as the myopic component and the intertemporal hedging component. The myopic component depends only on investors' mean variance preference on risky assets, not on the elasticity of intertemporal substitution, and therefore the myopic component will be the same no matter if the elasticity of intertemporal substitution is restricted to one or not. On the other hand, the intertemporal hedging component is dependent on the elasticity of intertemporal substitution, and the investor's intertemporal hedging demand on the optimal allocation is $(1-\frac{1}{\gamma^i})(\frac{\widetilde{Q}_1^i+\widetilde{Q}_2^i\,\mu_{et}}{(\gamma^i-1)})(\mathbf{V}^i\,\mathbf{V}^{i\prime})^{-1}\mathbf{V}^i\boldsymbol{\rho}\,\sigma_{\mu_e}$. When the elasticity of intertemporal substitution is not restricted to one, it is an affine function of the time-varying expected rates of changes in exchange rates with coefficients $\widetilde{Q}_1^i$ and $\widetilde{Q}_2^i$, $i\in[D,F]$. Here, $\widetilde{Q}_1^i+\widetilde{Q}_2^i\,\mu_{et}$ of the intertemporal hedging demand can be furthered separated into two parts: the intercept of the intertemporal hedging component $\widetilde{Q}_1^i$, and the pure changes in the time-varying expected rates of changes of exchange rates $\widetilde{Q}_2^i\,\mu_{et}$.

In the general case of $\gamma^i>1$, the value function J is maximized only with the solution that $\widetilde{Q}_2^i<0$, and the intercept $\widetilde{Q}_1^i<0$ is therefore negative. Recall that the optimal intertemporal hedging depends on the product of the coefficient of risk aversion γ^i, the variance-covariance matrix $(\mathbf{V}^i\,\mathbf{V}^{i\prime})$, the covariance of risky assets and states $\mathbf{V}^i\boldsymbol{\rho}\,\sigma_{\mu_e}$, and the instantaneous rates of changes of the value function $\widetilde{Q}_1^i+\widetilde{Q}_2^i\,\mu_{et}$. Since $\widetilde{Q}_2^i<0$, $\widetilde{Q}_1^i<0$, when $\gamma^i>1$, it implies that the investor will have a positive intertemporal hedging demand as the covariance between unexpected asset returns and revisions in expected future rates of changes in the exchange rates is negative $\mathbf{V}^i\boldsymbol{\rho}\,\sigma_{\mu_e}<0$, or a negative intertemporal hedging demand when $\mathbf{V}^i\boldsymbol{\rho}\,\sigma_{\mu_e}>0$.

In the case of $\mathbf{V}^i\boldsymbol{\rho}\,\sigma_{\mu_e}<0$, a negative sign implies that asset returns will be higher when expected future rates of changes in the exchange rates fall, while $\mathbf{V}^i\boldsymbol{\rho}\,\sigma_{\mu_e}>0$ implies that asset returns yield a lower one when the expected future exchange rates fall. Since the investor is normally long in domestic and foreign risky assets, a decline in expected future rates of changes in the exchange rates will improve or deteriorate the investment opportunity set and this depends on the sign of the covariance between risky assets and state. When the investment opportunity set deteriorates, there are offsetting considerations that determine an investor's attitudes toward assets that pay off. A highly risk averse investor ($\gamma^i > 1$) would like to hold assets that deliver wealth in unfavorable states of the world, because a risky asset can be thought as a valuable hedging instrument for conservative investors to hedge investment-opportunity risk coming from changes in exchange rates. In addition, a highly risk averse investor would not hold risky assets in favorable states of the world since risky assets provide no hedging ability.

We would like to know next whether the optimal intertemporal hedging demand can be affected by the mean-reversion phenomenon in the foreign exchange market. From the previous section, we have shown that $\widetilde{Q}_1^i + \widetilde{Q}_2^i \mu_{et}$ has a negative impact on the intertemporal hedging components when the investor is highly risk averse $\gamma^i > 1$. In this section, we would like to discuss the change of the intensity of the mean reversion effects κ on the foreign exchange rates over the intertemporal hedging demand. A comparative analysis of the mean reversion effect κ on the intertemporal hedging demand is equal to analyzing κ on $\widetilde{Q}_2^i$, since only $\widetilde{Q}_2^i$ of the intertemporal hedging component conveys information regarding the mean reversion intensity. Therefore, in Appendix C we show that a positive sign of the first derivative $\frac{\partial \widetilde{Q}_2^i}{\partial \kappa} > 0$ is derived for a highly risk averse investor $\gamma^i > 1$. A decline in the mean reversion speed of the expected future exchange rates will lead to a decline in $\widetilde{Q}_2^i$, because $\frac{\partial \widetilde{Q}_2^i}{\partial \kappa} > 0$. When the intensity of mean reversion κ decreases, the absolute value of $\widetilde{Q}_2^i$ increases since $\widetilde{Q}_2^i$ is negative in sign. Therefore, the absolute value of $\widetilde{Q}_1^i + \widetilde{Q}_2^i \mu_{et}$ increases when κ decreases. Additionally, when $\mathbf{V}^i \boldsymbol{\rho}\, \sigma_{\mu_e} < 0$, the intertemporal hedging demand of the investor is positive, and a decrease in κ reveals that the positive hedging ability of the risky asset is increased for a conservative investor.

When $\mathbf{V}^i \boldsymbol{\rho} \sigma_{\mu_e} > 0$, the intertemporal hedging demand of the investor is negative, and an increase in κ reveals that the negative hedging ability of the risky asset is increased for a conservative investor. Therefore, a decline in the mean reversion speed of the expected future exchange rate is equivalent to an increase in the persistence of shocks to expected future exchange rates, thus leading to an increase in the magnitude of intertemporal hedging demand coming from pure changes in expected future exchange rates. It is very straightforward that in this situation, the risky assets will provide more valuable hedging ability for a conservative investor as a negative sign for the covariance between unexpected asset returns and revisions in expected future rates of changes in exchange rates.

4. **Numerical analysis**

In this section we first present the relationship between optimal portfolio weights and the coefficient of relative risk aversion. Next, we focus on the determinants of the optimal intertemporal hedging component: the coefficient of risk-aversion γ^i, the elasticity of intertemporal substation φ^i, and the intensity of mean-reversion κ. Since we have already briefly introduced the impact of the three parameters on intertemporal hedging demand, hereafter we present the scenario analysis on the coefficient of risk-aversion and the elasticity of intertemporal substation (γ^i and φ^i), the coefficient of risk-aversion and the intensity of mean-reversion (γ^i and κ), and the elasticity of intertemporal substation and the intensity of mean-reversion (φ^i and κ) in order to identify the sensitivity of the three parameters on the optimal intertemporal hedging demand.

4.1 Sensitivity analysis of the investor's risk attitude on the optimal portfolio weight

In the optimal portfolio rules shown by Equation (24), the value function J is maximized only with the solution that $\tilde{Q}_2^i < 0$ when $\gamma^i > 1$, and $\tilde{Q}_1^i + \tilde{Q}_2^i \mu_{et}$ has a negative impact on the intertemporal hedging demand. Therefore, the sign of optimal intertemporal hedging demand depends only on the sign of the covariance between unexpected asset returns and expected foreign exchange rate changes $\mathbf{V}^{i}\boldsymbol{\rho}\,\sigma_{\mu_e}$. Since the risky assets the domestic investor faces consist of the domestic risky stock of country D, the foreign risky stock of country F, and the foreign riskless bond of country F, the sign of $\mathbf{V}^{i}\boldsymbol{\rho}\,\sigma_{\mu_e}$ is separately estimated by the covariance between the unexpected return of each individual risky asset and the expected change of foreign exchange rate. The covariance between unexpected return of the domestic stock of country D and the expected change of the foreign exchange rate is estimated to be negative, however, the covariance between unexpected return of the foreign stock of country F expressed in the domestic currency and the expected change of exchange rate is estimated to be positive. The intuition behind this is that when the foreign exchange rate is expected to fall, the foreign currency is less valuable than ever, and the future investment opportunities for the domestic risky stock worsen, but for the foreign risky stock it improves. Domestic risky stocks yield a high return, however, foreign risky stocks of country F expressed in the domestic currency yield a lower return when future investment opportunities worsen. Thus, it becomes a more valuable hedging instrument for the conservative investor since the future investment opportunity improves as the exchange rate is expected to fall. On contrary, foreign risky stocks provide no hedging ability for the conservative investor to hedge opportunity risk. In the end, a conservative investor spends more wealth on the domestic risky stock and less wealth on the foreign risky stock of country F expressed in domestic currency. We then estimate that the covariance between unexpected return of foreign riskless bonds and the expected change of the foreign exchange rate is negative. A negative sign of covariance between unexpected return of foreign riskless bonds and expected change of the exchange rate implies that foreign riskless bonds tend to have a higher return when the expected exchange rate falls, and when an investor is holding a long position in foreign riskless bonds, a decline in the expected future exchange rate induces a deterioration in the investment opportunity. A highly risk adverse investor will want to hold foreign riskless bonds that deliver wealth in unfavorable states of the world, and therefore a conservative investor will spend more wealth on foreign riskless bonds since a foreign riskless bond becomes a more valuable hedging instrument in order to hedge investment opportunity risk. Foreign riskless bonds yield a higher return in the situation where the foreign exchange rate is expected to fall when the investor is highly conservative.

Our optimal dynamic asset allocation on financial assets can be represented by the summation of the myopic component and the intertemporal hedging component for a long-term period investor. Figures 1 through 3 show us the relationship between the attitude of risk aversion and the optimal portfolio weights. Since the covariance between unexpected domestic risky asset return and expected change of the foreign exchange rate is estimated to be negative, Figures 1 and 3 present that the myopic component decreases as the coefficient of risk aversion increases, while the intertemporal hedging component first rises then decreases as the investor is more risk averse. When $\gamma^i > 1$ under the estimation of negative $\mathbf{V}^i \boldsymbol{\rho}\, \sigma_{\mu_e}$, an investor wants to hold assets that deliver wealth in unfavorable states of the world, and the intertemporal hedging demand is therefore positive. The intertemporal hedging demand is not monotonic in the coefficient of risk aversion.

Moreover, a risk averse investor will set up the exposure limit to the position of risky assets. When an investor is getting extremely risk adverse, any position in risky assets will exceed her exposure limit, and therefore she will limit her exposure to the risky assets in all states of the world. The intertemporal hedging component is therefore decreased with the coefficient of risk aversion. On the other hand, the covariance between unexpected foreign risky asset return and expected change of the foreign exchange rate is estimated to be positive. The myopic component declines as the investor becomes more risk averse, but the intertemporal hedging component of Figure 2 first decreases and then increases in value as the investor become more conservative. When the investor becomes extremely conservative, the investor’s optimal weight on risky assets will decrease to zero. From this sensitivity analysis, we conclude that the magnitude of intertemporal hedging demand first rises and then falls with the coefficient of risk aversion.

4.2 Scenario analysis of investors’ risk attitude, elasticity of intertemporal substitution, and the mean reversion intensity on the optimal intertemporal hedging component

As we know, the intertemporal hedging component on the optimal portfolio is not monotonic in the coefficient of risk aversion. We further analyze whether the impact of the co-movement of the coefficient of relative risk aversion γ^i and the elasticity of intertemporal substitution φ^i for the domestic risky stock of country D, foreign risky stock of country F, and foreign riskless bond of country F in the domestic currency value is monotonic on the optimal intertemporal hedging demand. From Figures 4 to 6, the intertemporal hedging component is more volatile when the relative risk aversion changes, and the intertemporal substitution reveals a doubtful trend on the intertemporal hedging demand. Therefore, the relative risk aversion is more sensitive than the elasticity of intertemporal substitution in determining the optimal weight on intertemporal hedging demand, however, the magnitude of the intertemporal hedging demand first rises and then decreases as the coefficient of risk aversion increases.

We next leave the coefficient of relative risk aversion be some specific value of 2 to 8, and try to understand the impact of the intensity of mean-reversion on the optimal intertemporal hedging component. From Figures 7 to 9, in the case where the coefficient of relative risk aversion is 2 in value, the intertemporal hedging demand decreases heavily as the intensity of mean-reversion increases. Moreover, if the coefficient of relative risk aversion is increased from 2 to 8, the intertemporal hedging demand decreases smoothly as the intensity of mean-reversion increases. Hence, the lower γ^i is, the higher the variation will be on intertemporal hedging demand as κ changes. We conclude that, as the intensity of mean reversion κ increases, the decline in intertemporal hedging demand increases as risk aversion decreases.

We now present the scenario analysis of the optimal weight of the intertemporal hedging demand when the coefficient of relative risk aversion γ^i and the intensity of mean-reversion κ change simultaneously. From Figures 10 to 12, as the intensity of mean reversion and the coefficient of risk aversion increase, the intertemporal hedging demand decreases. Both the coefficient of risk aversion and the intensity of mean reversion induce a negative impact on the intertemporal hedging demand, since the parameters of κ and γ^i range from 0 to 1, and 1 to 8 in our numerical analysis. The difference in parameter ranges causes the result that the relative risk aversion is more sensitive than the intensity of mean reversion in determining the optimal allocation on intertemporal hedging demand. Once we have narrowed down the range of coefficient of risk aversion, the sensitivity of the intensity of mean reversion on the optimal intertemporal hedging demand will increase.

5. Conclusions

There is a well-established literature which has found significant evidence that real exchange rates are mean-reverting, especially at long horizons. Central bank intervention may cause the presence of the mean-reverting phenomenon in foreign exchange rate markets. Most research extends intertemporal asset pricing models, with constant parameters, to an international setting. However, these papers commonly ignore the significant empirical findings on the mean-reverting phenomenon of exchange rates in the real world. Our goal has been to formally evaluate and quantify the mean-reverting effect of exchange rates into an international intertemporal model in order to find the optimal asset allocation strategies. Therefore, our model is more generalized than most international models, by inducing hedging strategies for time-varying investment opportunities in the spirit of the models tested by Dumas and Solnik (1992) and Harvey, Solnik and Zhou (1992).

By observing the limitation on log preference, we have presented a continuous-time recursive preference model which allows us not only to analyze the effect of risk aversion on investors' asset allocation decision as described by Zapatero (1995), but also to realize the intertemporal substitution effect in consumption. Our model accounts for myopic hedging as well as intertemporal hedging strategies, and furthermore our model allows for differences in beliefs across the representative agents by way of time-varying investment opportunities sets. This generalization allows us to analyze the parameter's effect on optimal allocations.

We derive the explicit solution by a more intuitive perturbation method of approximation around a particular point in the preference space, whereby the intertemporal elasticity equals 1. The numerical exercise shows that the optimal asset allocations can be dividend into a myopic component and an intertemporal hedging component, and both components are increased as the coefficient of risk aversion decreases. Moreover, the magnitude of intertemporal hedging demand first rises and then falls with an increase in the coefficient of risk aversion. When the coefficient of risk aversion, the intensity of mean reversion, and the elasticity of intertemporal substitution increase, the magnitude of intertemporal hedging demand decreases.

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Appendix A: The special solution where the elasticity of intertemporal substitution is restricted to one

In the special case where the elasticity of intertemporal substitution is restricted to one, a solution to the ordinary differential equation is $I^i(\mu_{et}) = \exp[\hat{Q}_0^i + \hat{Q}_1^i \mu_{et} + \frac{1}{2}\hat{Q}_2^i(\mu_{et})^2], i \in [D,F]$, and the three recursive equations of $\hat{Q}_2^D$, $\hat{Q}_1^D$ and $\hat{Q}_0^D$ result from collecting terms in ${\mu_{et}}^2$, μ_{et}, and constant terms for the representative investor in country D.

The equation that results from collecting terms in ${\mu_{et}}^2$ is:

$$\begin{aligned}
&\frac{1}{2}(\hat{Q}_2^D)^2(\frac{1}{\gamma^D}\sigma_{\mu_e}^2\rho'\rho + \frac{1}{1-\gamma^D}\sigma_{\mu_e}^2) + \hat{Q}_2^D\{-\frac{1}{2}\frac{1}{1-\gamma^D}\beta^D - \frac{1}{1-\gamma^D}\kappa \\
&+\frac{1}{\gamma^D}[\sigma_D'\rho\,\sigma_{\mu_e}(v_{21}^D + v_{31}^D) + (\sigma_F + \sigma_e)'\rho\,\sigma_{\mu_e}(v_{22}^D + v_{32}^D) + \sigma_e'\rho\,\sigma_{\mu_e}(v_{23}^D + v_{33}^D)]\} \\
&+\frac{1}{2}\frac{1}{\gamma^D}(v_{22}^D + v_{23}^D + v_{32}^D + v_{33}^D) = 0
\end{aligned} \tag{A1}$$

The equation that results from collecting terms in μ_{et} is:

$$\begin{aligned}
&\hat{Q}_1^D\{-\frac{1}{1-\gamma^D}\beta^D - \frac{1}{1-\gamma^D}\kappa + \frac{1}{\gamma^D}[v_{21}^D\sigma_D'\rho\,\sigma_{\mu_e} + v_{22}^D(\sigma_F + \sigma_e)'\rho\,\sigma_{\mu_e} + v_{23}^D\sigma_e'\rho\,\sigma_{\mu_e}] \\
&+\frac{1}{\gamma^D}[v_{31}^D\sigma_D'\rho\,\sigma_{\mu_e} + v_{32}^D(\sigma_F + \sigma_e)'\rho\,\sigma_{\mu_e} + v_{33}^D\sigma_e'\rho\,\sigma_{\mu_e}]\} + \hat{Q}_2^D\{\frac{1}{1-\gamma^D}\kappa\theta \\
&+\frac{1}{\gamma^D}(\mu_D - r^D)[v_{11}^D\sigma_D'\rho\,\sigma_{\mu_e} + v_{12}^D(\sigma_F + \sigma_e)'\rho\,\sigma_{\mu_e} + v_{13}^D\sigma_e'\rho\,\sigma_{\mu_e}] \\
&+\frac{1}{\gamma^D}(\mu_F + \sigma_F'\sigma_e - r^D)[v_{21}^D\sigma_D'\rho\,\sigma_{\mu_e} + v_{22}^D(\sigma_F + \sigma_e)'\rho\,\sigma_{\mu_e} + v_{23}^D\sigma_e'\rho\,\sigma_{\mu_e}] \\
&+\frac{1}{\gamma^D}(r^F - r^D)[v_{31}^D\sigma_D'\rho\,\sigma_{\mu_e} + v_{32}^D(\sigma_F + \sigma_e)'\rho\,\sigma_{\mu_e} + v_{33}^D\sigma_e'\rho\,\sigma_{\mu_e}]\} \\
&+\hat{Q}_1^D\hat{Q}_2^D\{\frac{1}{\gamma^D}\sigma_{\mu_e}^2\rho'\rho + \frac{1}{1-\gamma^D}\sigma_{\mu_e}^2\} + \frac{1}{2}\frac{1}{\gamma^D}\{(\mu_D - r^D)(v_{12}^D + v_{13}^D + v_{21}^D + v_{31}^D) \\
&+(\mu_F + \sigma_F'\sigma_e - r^D)(v_{23}^D + 2v_{22}^D + v_{32}^D) + (r^F - r^D)(v_{23}^D + 2v_{33}^D + v_{32}^D)\} = 0
\end{aligned} \tag{A2}$$

The equation that results from collecting terms in constant terms is:

$$\begin{aligned}
&(\hat{Q}_1^D)^2(\frac{1}{2}\frac{1}{\gamma^D}\sigma_{\mu_e}^2\rho'\rho + \frac{1}{2}\frac{1}{1-\gamma^D}\sigma_{\mu_e}^2) + \hat{Q}_2^D(\frac{1}{2}\frac{1}{1-\gamma^D}\sigma_{\mu_e}^2) + \hat{Q}_1^D\{\frac{1}{1-\gamma^D}\kappa\theta + \\
&\frac{1}{\gamma^D}(\mu_D - r^D)[v_{11}^D\sigma_D'\rho\,\sigma_{\mu_e} + v_{12}^D(\sigma_F + \sigma_e)'\rho\,\sigma_{\mu_e} + v_{13}^D\sigma_e'\rho\,\sigma_{\mu_e}] \\
&+\frac{1}{\gamma^D}(\mu_F + \sigma_F'\sigma_e - r^D)[v_{21}^D\sigma_D'\rho\,\sigma_{\mu_e} + v_{22}^D(\sigma_F + \sigma_e)'\rho\,\sigma_{\mu_e} + v_{23}^D\sigma_e'\rho\,\sigma_{\mu_e}] \\
&+\frac{1}{\gamma^D}(r^F - r^D)[v_{31}^D\sigma_D'\rho\,\sigma_{\mu_e} + v_{32}^D(\sigma_F + \sigma_e)'\rho\,\sigma_{\mu_e} + v_{33}^D\sigma_e'\rho\,\sigma_{\mu_e}]\} + \hat{Q}_0^D[-\frac{1}{1-\gamma^D}(\beta^D)] \\
&+\{\beta^D\log(\beta^D) - \beta^D + r^D + \frac{1}{2}\frac{1}{\gamma^D}[v_{11}^D(\mu_D - r^D)^2 + v_{22}^D(\mu_F + \sigma_F'\sigma_e - r^D)^2 \\
&+v_{33}^D(r^F - r^D)^2 + (v_{12}^D + v_{21}^D)(\mu_D - r^D)(\mu_F + \sigma_F'\sigma_e - r^D) + (v_{13}^D + v_{31}^D)(\mu_D - r^D)(r^F - r^D) \\
&+(v_{23}^D + v_{32}^D)(\mu_F + \sigma_F'\sigma_e - r^D)(r^F - r^D)]\} = 0
\end{aligned} \tag{A3}$$

Here, v_{rc}^D is the element of the rth row and cth column of matrix $(\mathrm{V}^D\mathrm{V}^{D'})^{-1}$ as shown below:

$$(\mathrm{V^D V^D}')^{-1} = \begin{bmatrix} \sigma_D'\sigma_D & \sigma_D'(\sigma_F+\sigma_e) & \sigma_D'\sigma_e \\ (\sigma_F+\sigma_e)'\sigma_D & (\sigma_F+\sigma_e)'(\sigma_F+\sigma_e) & (\sigma_F+\sigma_e)'\sigma_e \\ \sigma_e'\sigma_D & \sigma_e'(\sigma_F+\sigma_e) & \sigma_e'\sigma_e \end{bmatrix}^{-1} \tag{A4}$$

$$\equiv \begin{bmatrix} v_{11}^D & v_{12}^D & v_{13}^D \\ v_{21}^D & v_{22}^D & v_{23}^D \\ v_{31}^D & v_{32}^D & v_{33}^D \end{bmatrix}$$

We further note that:

$$\hat{Q}_2^D = \frac{-b \pm \sqrt{b^2-4ac}}{2a}, \tag{A5}$$

where

$$a = \frac{1}{2}\left(\frac{1}{\gamma^D}\sigma_{\mu_e}^2\rho'\rho + \frac{1}{1-\gamma^D}\sigma_{\mu_e}^2\right)$$

$$b = \{-\frac{1}{2}\frac{1}{1-\gamma^D}\beta^D - \frac{1}{1-\gamma^D}\kappa + \frac{1}{\gamma^D}[\sigma_D'\rho\,\sigma_{\mu_e}(v_{21}^D+v_{31}^D) + (\sigma_F+\sigma_e)'\rho\,\sigma_{\mu_e}(v_{22}^D+v_{32}^D) + \sigma_e'\rho\,\sigma_{\mu_e}(v_{23}^D+v_{33}^D)]\}$$

$$c = \frac{1}{2}\frac{1}{\gamma^D}(v_{22}^D+v_{23}^D+v_{32}^D+v_{33}^D)$$

Similarly, we have the following three equations of $\hat{Q}_2^F$, $\hat{Q}_1^F$ and $\hat{Q}_0^F$ for the representative investor in country F.

The equation that results from collecting terms in $(\mu_{et})^2$ is:

$$\begin{aligned} &\frac{1}{2}(\hat{Q}_2^F)^2\left(\frac{1}{\gamma^F}\sigma_{\mu_e}^2\rho'\rho + \frac{1}{1-\gamma^F}\sigma_{\mu_e}^2\right) + \hat{Q}_2^F\{-\frac{1}{2}\frac{\beta^F}{1-\gamma^F} - \frac{1}{1-\gamma^F}\kappa \\ &-\frac{1}{\gamma^F}[(\sigma_D-\sigma_e)'\rho\,\sigma_{\mu_e}(v_{11}^F+v_{31}^F) + \sigma_F'\rho\,\sigma_{\mu_e}(v_{12}^F+v_{32}^F) - \sigma_e'\rho\,\sigma_{\mu_e}(v_{13}^F+v_{33}^F)]\} \\ &+\frac{1}{2}\frac{1}{\gamma^F}(v_{11}^F+v_{13}^F+v_{31}^F+v_{33}^F) = 0 \end{aligned} \tag{A6}$$

The equation that results from collecting terms in μ_{et} is:

$$\begin{aligned} &\hat{Q}_1^F\{-\frac{1}{1-\gamma^F}(\beta^F) - \frac{1}{1-\gamma^F}\kappa - \frac{1}{\gamma^F}[(v_{11}^F+v_{31}^F)(\sigma_D-\sigma_e)'\rho\,\sigma_{\mu_e} + (v_{12}^F+v_{32}^F)\sigma_F'\rho\,\sigma_{\mu_e} \\ &-(v_{13}^F+v_{33}^F)\sigma_e'\rho\,\sigma_{\mu_e}]\} + \tilde{Q}_2^F\{\frac{1}{1-\gamma^F}\kappa\theta + \frac{1}{\gamma^F}(\mu_D + \sigma_e'\sigma_e - \sigma_D'\sigma_e - r^F)[v_{11}^F(\sigma_D-\sigma_e)'\rho\,\sigma_{\mu_e} \\ &+v_{12}^F\sigma_F'\rho\,\sigma_{\mu_e} - v_{13}^F\sigma_e'\rho\,\sigma_{\mu_e}] + \frac{1}{\gamma^F}(\mu_F - r^F)[v_{21}^F(\sigma_D-\sigma_e)'\rho\,\sigma_{\mu_e} + v_{22}^F\sigma_F'\rho\,\sigma_{\mu_e} - v_{23}^F\sigma_e'\rho\,\sigma_{\mu_e}] \\ &+\frac{1}{\gamma^F}(r^D - r^F + \sigma_e'\sigma_e)[v_{31}^F(\sigma_D-\sigma_e)'\rho\,\sigma_{\mu_e} + v_{32}^F\sigma_F'\rho\,\sigma_{\mu_e} - v_{33}^F\sigma_e'\rho\,\sigma_{\mu_e}]\} \\ &+\hat{Q}_1^F\hat{Q}_2^F\{\frac{1}{\gamma^F}\sigma_{\mu_e}^2\rho'\rho + \frac{1}{1-\gamma^F}\sigma_{\mu_e}^2\} - \frac{1}{2}\frac{1}{\gamma^F}\{(\mu_D + \sigma_e'\sigma_e - \sigma_D'\sigma_e - r^F)(v_{13}^F + 2v_{11}^F + v_{31}^F) \\ &+(\mu_F - r^F)(v_{12}^F+v_{21}^F+v_{23}^F+v_{32}^F) + (r^D - r^F + \sigma_e'\sigma_e)(v_{13}^F + 2v_{33}^F + v_{31}^F)\} = 0 \end{aligned} \tag{A7}$$

The equation that results from collecting terms in constant terms is:

$$
\begin{aligned}
&(\hat{Q}_1^F)^2(\frac{1}{2}\frac{1}{\gamma^F}\sigma_{\mu_e}^2\rho'\rho+\frac{1}{2}\frac{1}{1-\gamma^F}\sigma_{\mu_e}^2)+\hat{Q}_2^F(\frac{1}{2}\frac{1}{1-\gamma^F}\sigma_{\mu_e}^2)+\hat{Q}_1^F\{\frac{1}{1-\gamma^F}\kappa\theta \\
&+\frac{1}{\gamma^F}(\mu_D+\sigma_e'\sigma_e-\sigma_D'\sigma_e-r^F)[v_{11}^F(\sigma_D-\sigma_e)'\rho\,\sigma_{\mu_e}+v_{12}^F\sigma_F'\rho\,\sigma_{\mu_e}-v_{13}^F\sigma_e'\rho\,\sigma_{\mu_e}] \\
&+\frac{1}{\gamma^F}(\mu_F-r^F)[v_{21}^F(\sigma_D-\sigma_e)'\rho\,\sigma_{\mu_e}+v_{22}^F\sigma_F'\rho\,\sigma_{\mu_e}-v_{23}^F\sigma_e'\rho\,\sigma_{\mu_e}] \\
&+\frac{1}{\gamma^F}(r^D-r^F+\sigma_e'\sigma_e)[v_{31}^F(\sigma_D-\sigma_e)'\rho\,\sigma_{\mu_e}+v_{32}^F\sigma_F'\rho\,\sigma_{\mu_e}-v_{33}^F\sigma_e'\rho\,\sigma_{\mu_e}]\} \\
&+\hat{Q}_0^F[-\frac{1}{1-\gamma^F}(\beta^F)]+\{\beta^F\log(\beta^F)-\beta^F+r^F+\frac{1}{2}\frac{1}{\gamma^F}[v_{11}^F(\mu_D+\sigma_e'\sigma_e-\sigma_D'\sigma_e-r^F)^2 \\
&+v_{22}^F(\mu_F-r^F)^2+v_{33}^F(r^D-r^F+\sigma_e'\sigma_e)^2+(v_{12}^F+v_{21}^F)(\mu_F-r^F)(\mu_D+\sigma_e'\sigma_e-\sigma_D'\sigma_e-r^F) \\
&+(v_{13}^F+v_{31}^F)(\mu_D+\sigma_e'\sigma_e-\sigma_D'\sigma_e-r^F)(r^D-r^F+\sigma_e'\sigma_e) \\
&+(v_{23}^F+v_{32}^F)(r^D-r^F+\sigma_e'\sigma_e)(\mu_F-r^F)]\}=0
\end{aligned} \tag{A8}
$$

Here, v_{rc}^F is the element of the rth row and cth column of the matrix $(\mathrm{V}^F\mathrm{V}^{F'})^{-1}$.

Appendix B: The general solution where the elasticity of intertemporal substitution is not restricted to one

In the general case where the elasticity of intertemporal substitution is not restricted to one, a solution to the ordinary differential equation is written and shown by Equation (22) in section 3 as:

$$I^i(\mu_{et})=\exp[\widetilde{Q}_0^i+\widetilde{Q}_1^i\mu_{et}+\frac{1}{2}\widetilde{Q}_2^i(\mu_{et})^2],i\in[D,F]. \qquad \text{(22) (in section 3)}$$

$\widetilde{Q}_2^D$, $\widetilde{Q}_1^D$ and $\widetilde{Q}_0^D$ result from collecting terms in $(\mu_{et})^2$, μ_{et}, and the constant terms for the representative investor in country D, which can be individually expressed as follows.

The equation that results from collecting terms in $(\mu_{et})^2$ is:

$$
\begin{aligned}
&\frac{1}{2}(\widetilde{Q}_2^D)^2(\frac{1}{\gamma^D}\sigma_{\mu_e}^2\rho'\rho+\frac{1}{1-\gamma^D}\sigma_{\mu_e}^2)+\widetilde{Q}_2^D\{-\frac{1}{2}\frac{(\beta^D)^{\varphi^D}}{1-\gamma^D}-\frac{1}{1-\gamma^D}\kappa+\frac{1}{\gamma^D}[\sigma_D'\rho\,\sigma_{\mu_e}(v_{21}^D+v_{31}^D) \\
&+(\sigma_F+\sigma_e)'\rho\,\sigma_{\mu_e}(v_{22}^D+v_{32}^D)+\sigma_e'\rho\,\sigma_{\mu_e}(v_{23}^D+v_{33}^D)]\}+\frac{1}{2}\frac{1}{\gamma^D}(v_{22}^D+v_{23}^D+v_{32}^D+v_{33}^D)=0
\end{aligned} \tag{B1}
$$

The equation that results from collecting terms in μ_{et} is:

$$
\begin{aligned}
&\widetilde{Q}_1^D\{-\frac{1}{1-\gamma^D}(\beta^D)^{\varphi D}-\frac{1}{1-\gamma^D}\kappa+\frac{1}{\gamma^D}[v_{21}^D\sigma_D'\rho\,\sigma_{\mu_e}+v_{22}^D(\sigma_F+\sigma_e)'\rho\,\sigma_{\mu_e}+v_{23}^D\sigma_e'\rho\,\sigma_{\mu_e}] \\
&+\frac{1}{\gamma^D}[v_{31}^D\sigma_D'\rho\,\sigma_{\mu_e}+v_{32}^D(\sigma_F+\sigma_e)'\rho\,\sigma_{\mu_e}+v_{33}^D\sigma_e'\rho\,\sigma_{\mu_e}]\}+\widetilde{Q}_2^D\{\frac{1}{1-\gamma^D}\kappa\theta \\
&+\frac{1}{\gamma^D}(\mu_D-r^D)[v_{11}^D\sigma_D'\rho\,\sigma_{\mu_e}+v_{12}^D(\sigma_F+\sigma_e)'\rho\,\sigma_{\mu_e}+v_{13}^D\sigma_e'\rho\,\sigma_{\mu_e}] \\
&+\frac{1}{\gamma^D}(\mu_F+\sigma_F'\sigma_e-r^D)[v_{21}^D\sigma_D'\rho\,\sigma_{\mu_e}+v_{22}^D(\sigma_F+\sigma_e)'\rho\,\sigma_{\mu_e}+v_{23}^D\sigma_e'\rho\,\sigma_{\mu_e}] \\
&+\frac{1}{\gamma^D}(r^F-r^D)[v_{31}^D\sigma_D'\rho\,\sigma_{\mu_e}+v_{32}^D(\sigma_F+\sigma_e)'\rho\,\sigma_{\mu_e}+v_{33}^D\sigma_e'\rho\,\sigma_{\mu_e}]\} \\
&+\widetilde{Q}_1^D\widetilde{Q}_2^D\{\frac{1}{\gamma^D}\sigma_{\mu_e}^2\rho'\rho+\frac{1}{1-\gamma^D}\sigma_{\mu_e}^2\}+\frac{1}{2}\frac{1}{\gamma^D}\{(\mu_D-r^D)(v_{12}^D+v_{13}^D+v_{21}^D+v_{31}^D) \\
&+(\mu_F+\sigma_F'\sigma_e-r^D)(v_{23}^D+2v_{22}^D+v_{32}^D)+(r^F-r^D)(v_{23}^D+2v_{33}^D+v_{32}^D)\}=0
\end{aligned} \tag{B2}
$$

The equation that results from collecting terms in constant terms is:

$$(\widetilde{Q}_1^D)^2(\frac{1}{2}\frac{1}{\gamma^D}\sigma_{\mu_e}^2\rho'\rho+\frac{1}{2}\frac{1}{1-\gamma^D}\sigma_{\mu_e}^2)+\widetilde{Q}_2^D(\frac{1}{2}\frac{1}{1-\gamma^D}\sigma_{\mu_e}^2)+\widetilde{Q}_1^D\{\frac{1}{1-\gamma^D}\kappa\theta$$
$$+\frac{1}{\gamma^D}(\mu_D-r^D)[v_{11}^D\sigma_D'\rho\sigma_{\mu_e}+v_{12}^D(\sigma_F+\sigma_e)'\rho\sigma_{\mu_e}+v_{13}^D\sigma_e'\rho\sigma_{\mu_e}]$$
$$+\frac{1}{\gamma^D}(\mu_F+\sigma_F'\sigma_e-r^D)[v_{21}^D\sigma_D'\rho\sigma_{\mu_e}+v_{22}^D(\sigma_F+\sigma_e)'\rho\sigma_{\mu_e}+v_{23}^D\sigma_e'\rho\sigma_{\mu_e}] \tag{B3}$$
$$+\frac{1}{\gamma^D}(r^F-r^D)[v_{31}^D\sigma_D'\rho\sigma_{\mu_e}+v_{32}^D(\sigma_F+\sigma_e)'\rho\sigma_{\mu_e}+v_{33}^D\sigma_e'\rho\sigma_{\mu_e}]\}+\widetilde{Q}_0^D[-\frac{1}{1-\gamma^D}(\beta^D)^{\varphi^D}$$
$$+\{-\frac{1}{1-\varphi^D}(\beta^D)^{\varphi^D}+\frac{\varphi^D}{1-\varphi^D}\beta^D+r^D+\frac{1}{2}\frac{1}{\gamma^D}[v_{11}^D(\mu_D-r^D)^2+v_{22}^D(\mu_F+\sigma_F'\sigma_e-r^D)^2$$
$$+v_{33}^D(r^F-r^D)^2+(v_{12}^D+v_{21}^D)(\mu_D-r^D)(\mu_F+\sigma_F'\sigma_e-r^D)+(v_{13}^D+v_{31}^D)(\mu_D-r^D)(r^F-r^D)$$
$$+(v_{23}^D+v_{32}^D)(\mu_F+\sigma_F'\sigma_e-r^D)(r^F-r^D)]\}=0]$$

Similarly, we can derive another three equations of $\widetilde{Q}_2^F$, $\widetilde{Q}_1^F$ and $\widetilde{Q}_0^F$ for the representative investor in country F.

Appendix C: Comparative analysis of mean reversion intensity on the intertemporal hedging demand

In this appendix we would like to analyze the change of intertemporal hedging demand with respect to the mean reversion factor κ. From the optimal allocation on risky assets (shown by Equation (24)), the component of intertemporal hedging (which conveys information regarding mean reversion intensity) is $\widetilde{Q}_2^i$, and therefore this section briefly analyzes the change of $\widetilde{Q}_2^i$ with respect to the mean reversion factor κ**.** First, we present two quadratic equations for $\widetilde{Q}_2^D$ and $\widetilde{Q}_2^F$ as follows.

The quadratic equations for $\widetilde{Q}_2^D$ is:

$$\frac{1}{2}(\widetilde{Q}_2^D)^2(\frac{1}{\gamma^D}\sigma_{\mu_e}^2\rho'\rho+\frac{1}{1-\gamma^D}\sigma_{\mu_e}^2)+\widetilde{Q}_2^D\{-\frac{1}{2}\frac{(\beta^D)^{\varphi^D}}{1-\gamma^D}-\frac{1}{1-\gamma^D}\kappa+\frac{1}{\gamma^D}[\sigma_D'\rho\sigma_{\mu_e}(v_{21}^D+v_{31}^D)$$
$$+(\sigma_F+\sigma_e)'\rho\sigma_{\mu_e}(v_{22}^D+v_{32}^D)+\sigma_e'\rho\sigma_{\mu_e}(v_{23}^D+v_{33}^D)]\}+\frac{1}{2}\frac{1}{\gamma^D}(v_{22}^D+v_{23}^D+v_{32}^D+v_{33}^D)=0 \tag{C1}$$

The quadratic equations for $\widetilde{Q}_2^F$ is:

$$\frac{1}{2}(\widetilde{Q}_2^F)^2(\frac{1}{\gamma^F}\sigma_{\mu_e}^2\rho'\rho+\frac{1}{1-\gamma^F}\sigma_{\mu_e}^2)+\widetilde{Q}_2^F\{-\frac{1}{2}\frac{(\beta^F)^{\varphi^F}}{1-\gamma^F}-\frac{1}{1-\gamma^F}\kappa$$
$$+\frac{1}{\gamma^F}[\sigma_D'\rho\sigma_{\mu_e}(v_{21}^D+v_{31}^D)+(\sigma_F+\sigma_e)'\rho\sigma_{\mu_e}(v_{22}^D+v_{32}^D)+\sigma_e'\rho\sigma_{\mu_e}(v_{23}^D+v_{33}^D)]\} \quad . \tag{C2}$$
$$+\frac{1}{2}\frac{1}{\gamma^F}(v_{22}^D+v_{23}^D+v_{32}^D+v_{33}^D)=0$$

Let A, B, and C refer to the coefficients in equations (C1) and (C2) associated with $(\widetilde{Q}_2^i)^2$, $\widetilde{Q}_2^i$, and constant terms, respectively. We now obtain:

$$\phi(\widetilde{Q}_2^i)=A(\widetilde{Q}_2^i)^2+B\widetilde{Q}_2^i+C, i\in[D,F]. \tag{C3}$$

Totally differentiating equation (C3) with respect to κ yields:

$$\frac{\partial \widetilde{Q}_2^i}{\partial \kappa} = \frac{\partial \phi(\widetilde{Q}_2^i) \big/ \partial \kappa}{-\partial \phi(\widetilde{Q}_2^i) \big/ \partial \widetilde{Q}_2^i} > 0 \cdot$$

For a more risk-averse investor ($\gamma^i > 1$):

$$\partial \phi(\widetilde{Q}_2^i) \big/ \partial \kappa = -\frac{1}{1-\gamma^i} \widetilde{Q}_2^i < 0, \text{ (since } \widetilde{Q}_2^i < 0 \text{ when } \gamma^i > 1)$$

and

$$-\partial \phi(\widetilde{Q}_2^i) \big/ \partial \widetilde{Q}_2^i = -(2AQ_2 + B) = -\left[2A\left(\frac{-B + \sqrt{B^2 - 4AC}}{2A} \right) + B \right] = -\sqrt{B^2 - 4AC} < 0$$

This result is obtained by choosing the solution associated with the negative root of equations (B2) and (B3), i.e., choosing the positive root of the discriminant of the quadratic equations (B2) and (B3) as discussed earlier.

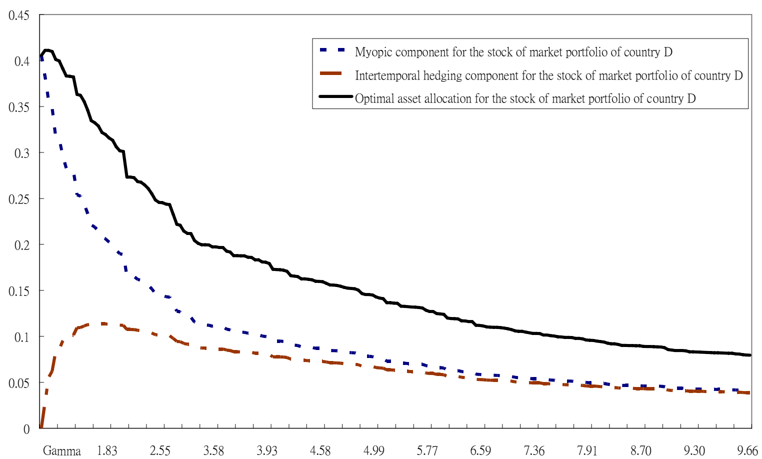

Figure 1. The optimal dynamic asset allocation on the stock of market portfolio of country D and their components in relation to the investor's coefficient of relative risk aversion (γ)

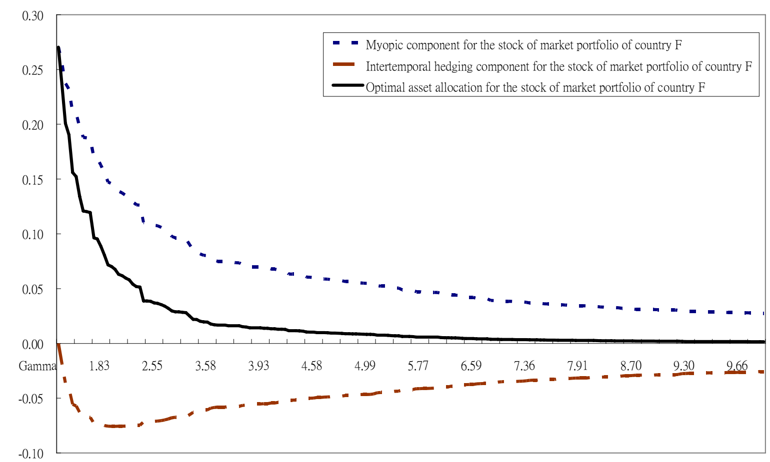

Figure 2. The optimal dynamic asset allocation on the stock of market portfolio of country F and their components in relation to the investor's coefficient of relative risk aversion (γ)

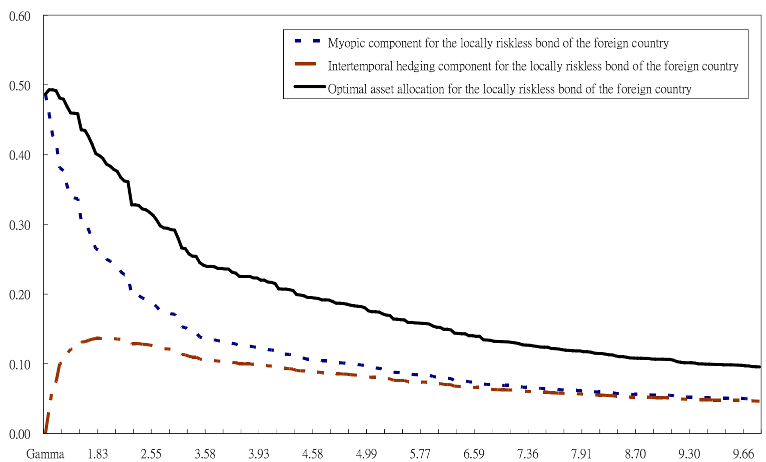

Figure 3. The optimal dynamic asset allocation on the local riskless bond of the foreign country in relation to the investor's coefficient of relative risk aversion (γ)

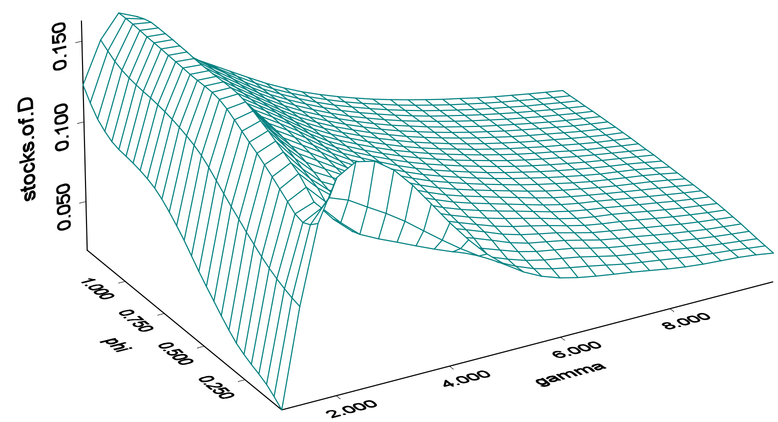

Figure 4. The intertemporal hedging component of the optimal dynamic asset allocation on the stock of the market portfolio of country D in relation to the investor's coefficient of relative risk aversion (γ) and elasticity of intertemporal substitution (φ)

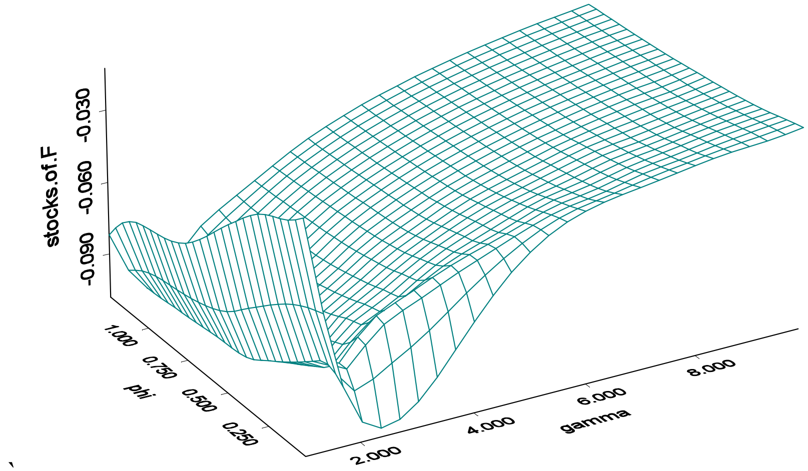

Figure 5. The intertemporal hedging component of the optimal dynamic asset allocation on the stock of the market portfolio of country F in relation to the investor's coefficient of relative risk aversion (γ) and elasticity of intertemporal substitution (φ)

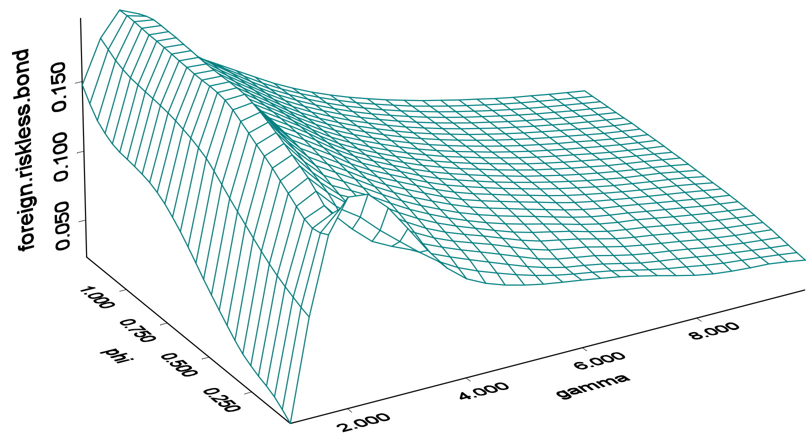

Figure 6. The intertemporal hedging component of the optimal dynamic asset allocation on the local riskless bond of the foreign country in relation to the investor's coefficient of relative risk aversion (γ) and elasticity of intertemporal substitution (φ)

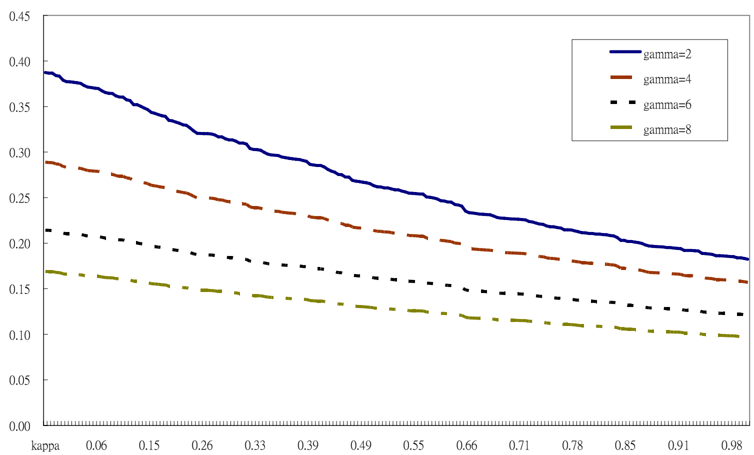

Figure 7. The intertemporal hedging component of the optimal dynamic asset allocation on the stock of the market portfolio of country D in relation to the intensity of mean reversion of the exchange rate (κ). This is with respect to different coefficient of relative risk aversion (γ)

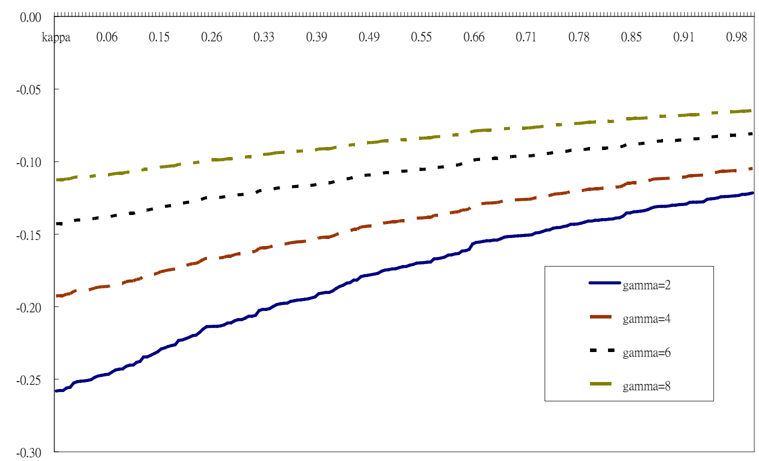

Figure 8. The intertemporal hedging component of the optimal dynamic asset allocation on the stock of the market portfolio of country F in relation to the intensity of mean reversion of the exchange rate (κ). This is with respect to different coefficient of relative risk aversion (γ)

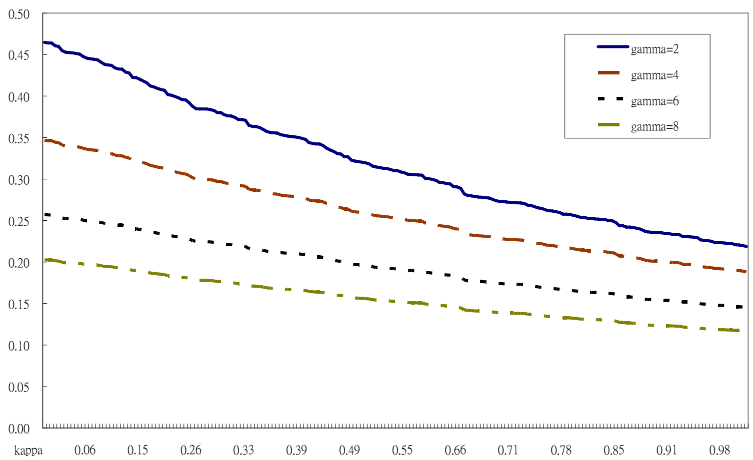

Figure 9. The intertemporal hedging component of the optimal dynamic asset allocation on the local riskless bond of the foreign country in relation to the intensity of mean reversion of the exchange rate (κ). This is with respect to different coefficient of relative risk aversion (γ)

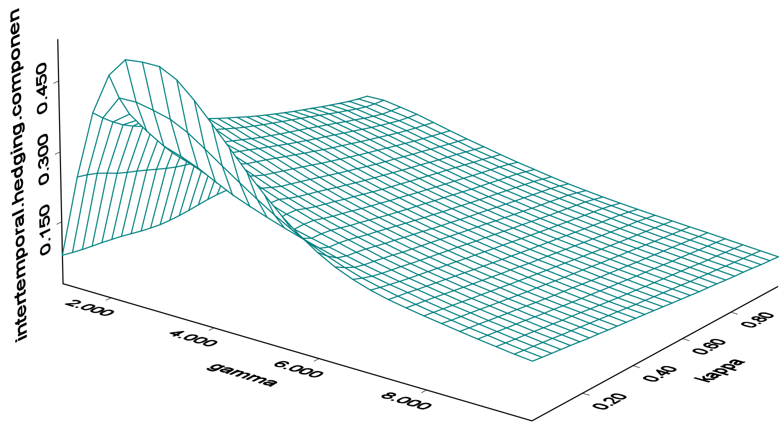

Figure 10. The intertemporal hedging component of the optimal dynamic asset allocation on the stock of the market portfolio of country D in relation to the intensity of mean reversion of the expected future exchange rate (κ) and the investor's coefficient of relative risk aversion (γ)

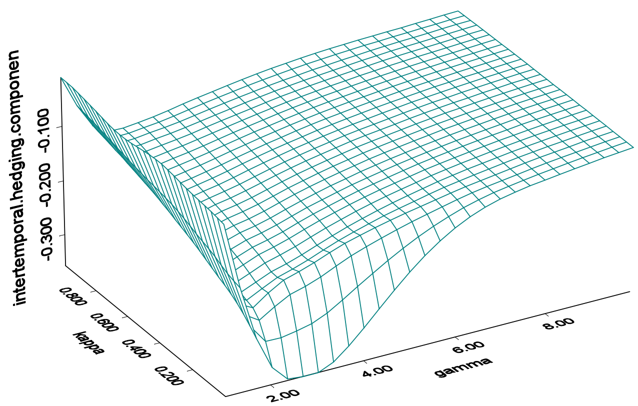

Figure 11. The intertemporal hedging component of the optimal dynamic asset allocation on the stock of the market portfolio of country F in relation to the intensity of mean reversion of the expected future exchange rate (κ) and the investor's coefficient of relative risk aversion (γ)

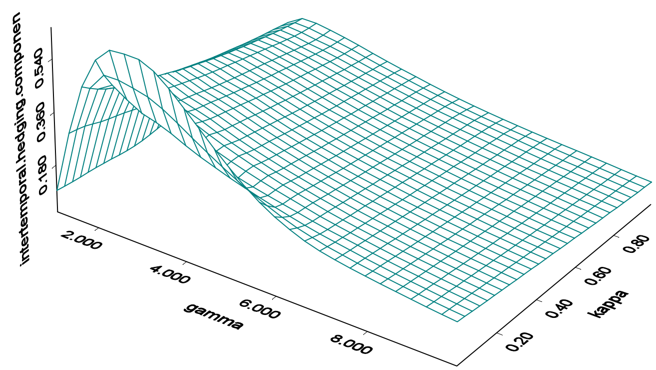

Figure 12. The intertemporal hedging component of the optimal dynamic asset allocation on the local riskless bond of the foreign country in relation to the intensity of mean reversion of the expected future exchange rate (κ) and the investor's coefficient of relative risk aversion (γ)