General-Equilibrium Pricing of Stock Index Futures with Regular and Irregular Stochastic Volatilities

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Abstract

This study highlights the effects of regular and irregular volatilities on futures pricing and derives a general-equilibrium formula in its closed form. Our comparative static and simulation results show that disputed arguments in literature among economic variables can be explained by different dimensions of market volatility in the economy. For example, the relative size of speed of mean reversion parameters in the dynamics of growth and variability features dominates the relationship between futures price and market volatility.

Keywords: jump risk, information time, information intensity, intertemporal futures pricing model.

JEL: D52, G13.

Ⅰ Introduction

This study explores whether returns volatility explains the value for stock futures, which simultaneously entails reciprocal rights and obligations. The setting of symmetric futures payoffs is in contrast with that for call or put options, for which spot price with greater volatility endows the holder a greater in-the-money probability within the Black and Scholes' (1973) framework. Namely, market volatility serves as a variable to option price but does not appear in the cost of carry model.

Following Hemler and Longstaff (1991), we posit and model that, nevertheless, market volatility affects futures price due to the fact that the economic variables may be highly interrelated in the general-equilibrium sense. We also model that the signs and magnitudes of the correlations among the variables vary among different types of market volatility. Specifically, a usual or persistent random source may be taken as a 'regular' volatility and specified by a stochastic process, whereas a rare and abrupt influential one may be regarded as a source of 'irregular' volatility and specified by a different process. [1] The coexistence of the two processes illuminates our understanding of the well-known leptokurtosis of financial return series in that observations with the regular one serve to form the relative high peak and those with the irregular one serve as the relative fat tails in terms of statistical distribution.

Namely, we can decompose volatility into the regular or irregular part in terms of information arrivals and accompanied effects. In a fixed time interval, information arriving with a tiny frequency but with a great impact such as the 911 terrorist attack in 2001 and the stock market crash in 1987 may be deemed irregular, whereas

[1] Merton (1976) uses the terms 'normal' and 'abnormal' vibrations, which are assumed to be a constant and a Poisson process separately, to decompose the total change in stock price in a 'partial-equilibrium' framework (pp.127, the second paragraph). His still and more well-known name of the stochastic 'abnormal' part is 'jump'. In this study, we follow Merton's line with applying the 'information time' setting on his 'normal' part to address the synchronic and stochastic essence of state variables in a 'general-equilibrium' framework. Thus, we use the terms 'regular' and 'irregular' instead of 'normal' and 'abnormal' to make a differentiation.

information occurring with a higher frequency and with a less significant effect may be regarded as regular. The frequency of the regular or irregular information arrivals is the so-called regular or irregular “information arriving intensity”. The great impact by an irregular information arrival is the so-called “impulse effect”.

Specifically, pricing contingent claims can be achieved by the no-arbitrage argument or a general equilibrium framework. The former is based on an equivalent comparison of expected returns and risks between different positions. In terms of the milestone of continuous-time intertemporal models, Black and Scholes’ (1973) option pricing theory, an arbitrage portfolio yields a return equivalent to a risk-free rate after adjusting weights and replicating components repeatedly and massively by market participants. The motive of chasing up arbitrage opportunities guarantees formulation of the well-known BS partial differential equation and the closed-form formulas. However, the no-arbitrage replication strategy needs marketable assets which may not be available in reality to eliminate market risks, which is the so-called market completeness issue [2]. Additional assumptions are needed to deal with the ‘incompleteness’ predicament in the no-arbitrage strand. [3] Moreover, dynamic linkages among market variables can not be comprehended. The employment of a general equilibrium model to price contingent claims is justified thereby. Cox, Ingersoll, and Ross (1985) have established such a standard.[4] The salient role of market volatility in futures pricing can be addressed through their well-known

[2] Please refer to Duffie (2001) for a more thorough discussion.

[3] For instance, Merton (1976) and Cox and Ross (1976) use assumptions of constant Sharpe ratio and diversifiable property to deal with the jump risk. The cost of these maintained hypotheses is that risk premia appear in fundamental valuation equations and resulting formulas (e.g., Cox, Ingersoll, and Ross (1985) and Hull and White (1987)). And the implementation of the pricing formulas in reality is hindered.

[4] The fact that the agent’s utility is unobservable may be a major drawback for applying general equilibrium models in contrast to the no-arbitrage argument. Yet Hemler and Longstaff (1991) derive a futures pricing formula and claim it is a “preference-free” model.

platform. For instance,[5] Hemler and Longstaff (1991) derive a futures pricing formula and find that market volatility plays a significant role in futures pricing under a general equilibrium framework. This result inspires our intention to further explore effects from different components of market volatility, namely, the regular and irregular parts.

Merton (1976) accentuates that a rare event such as the discovery of an important new oil well or the loss of a court triggers sudden and abrupt stock price changes of a specific industry or a firm. Yet the diffusion term and the Wiener process in the widely-used log-normal diffusion process are not adequate for characterizing the large resulting impulse and small occurrence probability of the jump. He thus appends a Poisson process to the original diffusion process in order to depict the abnormal effect. Subsequent empirical and theoretical studies follow him and highlight the event risk in pricing contingent claims. (For example, Jorion (1988), Naik and Lee (1990), Bates (1991, 1996, 2000), Bakshi, et al. (1997), Scott (1997), Hilliard and Reis (1998), Duffie, et al. (2000), Pan (2002), Eraker, Johannes, and Polson (2003), Eraker (2004), Liu, Longstaff, and Pan (2003)).

As the stochasticity of irregular market volatility dynamics can be described by the Poisson process, the stochasticity of regular part can be described by the so-called SV (stochastic volatility) models. Namely, a supplementary process is used to describe the stochastic property of the diffusion parameter embedded in the original spot price process. Prevalent formulations include the generalized Wiener process, squared-root process, Ornstein-Uhlenbeck processes, and their variations or

[5] There are various extensions of general-equilibrium futures pricing models from CIR's (1985) model. For instance, Richard and Sundaresan (1981) construct an intertemporal rational expectation model in a multi-good economy with identical consumers. They find that effectiveness of consumption hedging is the key to discern normal backwardation or Contango phenomenon. Cox, Ingersoll, and Ross (1981) prove that if futures price and bond price are positively correlated, then the futures price is less than the forward price; if they are negatively correlated, then the reverse result holds.

combinations. However, these models provide fitting flexibility at cost of numerous parameters to be estimated.[6] Chang, Chang, and Lim (1998) adopt Clark's (1973) terminology and develop a more parsimony model to manage the predicament. In contrast to the SV models, which include an additional process with more parameters to be estimated, a Bernoulli variable with a single parameter to be estimated is substituted into the spot return process. Chang, Chang, and Lim (1998) argue that the information-time setting not only helps to reduce modeling complexity but also to help to capture the synchronic variations across state variables.

Extant literature emphasizes the coexistence of stochastic volatility and jump settings to capture empirical phenomena, acknowledging that the pure regular setting may underestimate total volatility.[7] For instance, eight estimated standard deviations are needed to fully reflect effects from U.S. historical major events by Heston's (1993) stochastic volatility model. (Eraker, et al., 2003). Moreover, the coexisting stochastic volatility and jump settings may explain the 'leptokurtic' phenomenon. Namely, a normal distribution is inadequate to fit financial return series for their high-peaked, fat-tailed, or even skewed characteristics.[8] On the other hand, as a result of effects from regular and irregular volatility on futures pricing have not been discussed in previous literature through a general-equilibrium framework, we apply the information-time setting of Chang, et al. (1998) to describe regular stochastic volatility and Merton's (1976) jump setting to capture irregular stochastic volatility in

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6 Please refer to Hull and White (1987), Wiggins (1987), Scott (1987), Chesney and Scott (1989), Melino and Turnbull (1990), Stein and Stein (1991), Heston (1993), Amin and Ng (1993) for more details.

7 See Bakshi, et al. (1997), Bates (1996, 2000), Pan (2002), Eraker, et al. (2003), Liu, et al. (2003), and Earker (2004) for more details.

8 Various distributions are suggested to replace the normal assumption in static sense. For instance, Fama (1965) claims that stable Paretian distribution with characteristic exponent less than 2 could do; Paretz (1972) and Blattberg and Gonedes (1974) suggest student *t* distribution; as for Kon (1984), a discrete mixture of normal distributions is proposed to explain the observed patterns of significant kurtosis and positive skewness of daily data.

Cox, Ingersoll, and Ross' (1985) economy. Equilibrium stock index futures price and interest rate are derived in their analytic forms. Partial differentiation and simulation results are provided as well to investigate effects from different components of market volatility. We find that both sign and magnitudes of correlations among variables change substantially even when different economies are equipped with similar features of growth and variability outlooks. Decomposing volatility into regular- and irregular- parts, not only helps minimizing the complications but also identifying the different relationships among main variables, especially when the economic outlooks appear to be similar.

In summary, we develop a more generalized futures pricing model similar to Hemler and Longstaff (1991) by adopting the jump and the information-time settings. We find that the stochastic regular and irregular volatilities as well as the relative mean-reverting speed of the growth and variability features in the index dynamics serve as primary pricing factors. The remainder of this paper is organized as follows. We derive a close-formed pricing formula allowing for information-time and jump settings in Section II. Partial differentiation and simulation results are provided in Section III. Section IV concludes our study.

II Theoretical Framework and Futures Pricing Formulas

We develop an economy based on CIR (1985) and Hemler and Longstaff (1991) with stochastic state macroeconomic variables to depict dynamics of growth and variability features, and with an additional consideration of possible extreme events as Merton (1976) has noted. The information-time setting of Chang, et al. (1998) is employed as well to address the stochastic essence of the regular volatility. Not only for its parsimony as compared with other usual SV models, the information time setting is suitable for the general equilibrium model in that it highlights the notion that an information arrival simultaneously affects more than one state variable governing economic equilibrium conditions.

As shown in Equation (1), a Poisson process $k\,dm(t)$ is supplemented in an typical Itô process $\frac{dp}{p}$ to characterize the irregular volatility component of the production output *p*. k is a deterministic impulse function of percentage change of the production output level resulted from a jump event, $dm(t)$ is a Bernoulli variable controlling the occurrence of a jump event with probability $\lambda_m \times dt$ in an instantaneous time span dt. For instance, very few investors could accurately foresee the crash of 1987 before the "Black Monday" so that market anticipation on the occurrence probability $\lambda_m \times dt$ is negligible a priori. Nevertheless, the associated impulse effect *k* is tremendous because the S&P 500 dropped 20.4% on the single day.

$$\frac{dp}{p} = \left[\mu(p,t)\,X(p,t) - \lambda_m\,k\right]dt + \sigma(p,t)\sqrt{Y(p,t)}\,dz_{p(t)} + k\,dm(t), \quad (1)$$

where

$$\mu(p,t)\,X(p,t)\,dt = \mu(p,n)\,X(p,n)\,dn = \mu\,X(p,n)\,dn\,,$$

$$\sigma^2(p,t)\,Y(p,t)\,dt = \sigma^2(p,n)\,Y(p,n)\,dn = \sigma^2\,Y(p,n)\,dn\,,$$

$$dm(t) = \begin{cases} 1 & \text{with probability } \lambda_m \times dt \\ 0 & \text{with probability } 1 - \lambda_m \times dt\,, \end{cases}$$

and

$$dn(t) = \begin{cases} 1 & \text{with probability } \lambda_n \times dt \\ 0 & \text{with probability } 1 \text{-} \lambda_n \times dt\,. \end{cases}$$

Note that λ_m and λ_n are the so-called irregular and regular information intensities under the property of Poisson process, because their sizes are positively related to the frequencies of irregular and regular information arrivals given a fixed time interval. With more independent regular and irregular events occurring in the time interval, the spot price is more apt to change in an instantaneous time span. On the other hand, in contrast to a "calendar-time" span Δt which is with a constant time interval, an "information-time" span Δn is with random time interval in cases when the arrival timing of information is uncertain. The random variable dn is substituted into the growth feature $\mu(p,t)\,X(p,t)$ and the variability feature $\sigma(p,t)\sqrt{Y(p,t)}$ of the economy in order to replace the original calendar-time setting dt in the intertemporal sense. Namely, the "synchronic" stochasticity of the growth and variability features is controlled by the same Bernoulli variable dn in an instantaneous time span dt with probability $\lambda_n \times dt$. It is devised to reduce the modeling complexity and articulate the economic fact that no regular volatility is resulted in if no information arrives. In addition, μ and σ are constants in the information-time sense, and X and Y are two stochastic state variables affecting the production level through drift and diffusion terms in Equation (1). The dynamics may be described as

$$dX = a(X,t)\left[b(X,t) - X(X,t)\right]dt + c(X,t)\sqrt{X(X,t)}\,dz_{X(t)}\,,$$

$$\begin{aligned} &a(X,t)\left[b(X,t) - X(X,t)\right]dt \\ &\qquad = a(X,n)\left[b(X,n) - X(X,n)\right]dn = a(b-X)\,dn\,, \end{aligned}$$

$$c^2(X,t)\,X(X,t)\,dt = c^2(X,n)\,X(X,n)\,dn = c^2\;X\,dn\,,$$

$$dY = f(Y,t)\left[g(Y,t) - Y(Y,t)\right]dt + h(Y,t)\sqrt{Y(Y,t)}\,dz_{Y(t)}\,,$$

$$\begin{aligned} &f(Y,t)\left[g(Y,t) - Y(Y,t)\right]dt \\ &\qquad = f(Y,n)\left[g(Y,n) - Y(Y,n)\right]dn = f(g-Y)\,dn\,, \end{aligned}$$

and

$$h^2\left(Y,t\right)Y\left(Y,t\right)dt = h^2\left(Y,n\right)Y\left(Y,n\right)dn = h^2\ Y\ dn\ ,$$

where $\left(a, f\right)$ are constant and positive parameters of mean-reverting speed, $\left(b, g\right)$ are long-term averages, and $\left(c, h\right)$ are diffusion terms of $\left(X, Y\right)$ in the two Ornstein-Uhlenback processes with the square-root characteristic. Specifically, we generalize the setting of Hemler and Longstaff (1991) and CIR (1985) in order to discern effects from the growth and variability features in the production dynamics, which turns out to be relevant in Section III.

The calendar-time based stochastic processes can be stated in the information-time sense with straightforward algebraic manipulations:

$$\frac{dp}{p} = \mu\ X\left(n,t\right)\ dn\left(t\right) - \lambda_m\ k\ dt + \sigma\sqrt{Y\left(n,t\right)}\ dz_{p(n)} + k\ dm\left(t\right),$$

$$dX\left(n,t\right) = a\left[b - X\left(t\right)\right]dn\left(t\right) + c\sqrt{X\left(t\right)}\ dz_{X(n)}\ , \tag{2}$$

and

$$dY\left(n,t\right) = f\left[g - Y\left(t\right)\right]dn\left(t\right) + h\sqrt{Y\left(t\right)}\ dz_{Y(n)}\ ,$$

where $dz_{\bullet(n)} = \varepsilon_{\bullet}\left(t\right)\sqrt{dn\left(t\right)}$ is an information-time based Wiener process characterized by a standardized normal random variable $\varepsilon_{\bullet}\left(t\right)$ and $n\left(t\right)$. Note that the lowercase is denoted by *n* instead of *t*. Consistent to real world phenomenon, Equation (2) models that the sources of regular volatility, namely, both growth and variability features, fluctuate synchronically. For instance, overreaction to earning growth and excess volatility usually amplify each other in financial markets (Shiller, 2002). We also assume that $dm\left(t\right)$, $dn\left(t\right)$, $dz_{p(t)}$, $dz_{X(t)}$, and $dz_{Y(t)}$ are mutually independent innovations.

As in Hemler and Longstaff's (1991) setting, there are a fixed number of identical agents who seek to maximize their time-additive preferences. The representative agent's lifetime utility is of the form:

$$E_t \left[\int_t^{\infty} e^{-\rho s} \ln\left(C(t)\right) dt \right],$$

where $C(t)$ denotes the agent's consumption at time t, and ρ is agent's intertemporal discount rate of his/her lifetime utility. The investor chooses the levels of portfolio weights $w_{\square}$ to allocate his/her wealth W on the physical goods p, contingent claims F^i, riskless borrowing or lending positions, and his/her consumption in order to maximize the lifetime utility of the representative agent subject to a budget constraint:

$$dW = w_p \, W \frac{dp}{p} + \sum_i w_{F^i} \, W \frac{dF^i}{F^i} + \left(1 - w_p - \sum_i w_{F^i}\right) W \, r \, dt - C \, dt .$$

The value of stock or the underlying asset of the index futures contract is the realized wealth after consumption. Its dynamics ($\frac{dW}{W}$) and the equilibrium interest rate r can be obtained under the market clearing conditions and standard optimal control procedures.

$$\frac{dW}{W} = \mu \, X \, dn - \left(\lambda_m \, k + \rho\right) dt + \sigma \sqrt{Y} \, dz_{p(n)} + k \, dm . \tag{3}$$

$$r = \lambda_n \, \mu \, X - \lambda_m \, k - \lambda_n \, V . \tag{4}$$

It is intuitive that the equilibrium interest rate r is determined by three components including the expected return on the production activity μX, the expected jump effect $\lambda_m k$ from the irregular volatility component, and the regular volatility V where

$$V \equiv \sigma^2 \, Y . \tag{5}$$

The resulting dynamics of the equilibrium interest rate and the regular volatility can be obtained as well:

$$
\begin{aligned}
dr &= a\left\{\left[\lambda_n\, \mu\, b - \lambda_m\, k - \lambda_n\, V - \lambda_n\, \frac{f}{a}\left(\sigma^2 g - V\right)\right] - r\right\} dn \\
&+ \sqrt{\lambda_n\, \mu}\; c\, \sqrt{r + \lambda_m\, k + \lambda_n\, V}\; dz_{X(n)} - \lambda_n\, \sigma\, h \sqrt{V}\; dz_{Y(n)} ,
\end{aligned}
$$

and

$$dV = f\left(\sigma^2 g - V\right) dn + \sigma\, h \sqrt{V}\; dz_{Y(n)} .$$

Moreover, a partial differential equation for pricing the sock index futures contract is derived: [9]

$$
\begin{aligned}
& F_t + F_W\left[-\lambda_n\, \sigma^2\, Y\, W - W\left(\lambda_m\, k + \rho\right) + \lambda_n\, W\, \mu\, X\right] \\
& + F_X\, \lambda_n\, a\left(b - X\right) + F_Y\, \lambda_n\, f\left(g - Y\right) + \frac{1}{2} F_{WW}\, \lambda_n\, W^2\, \sigma^2\, Y \\
& + \frac{1}{2} F_{XX}\, \lambda_n\, c^2\, X + \frac{1}{2} F_{YY}\, \lambda_n\, h^2\, Y = r\, F .
\end{aligned} \tag{6}
$$

Nevertheless, *X* and *Y* are unobservable state variables. In order to express stock index futures price with economic meanings, we transforms the unobservable state variables X and Y into observable interest rate r and market variance V by Equation (4) and (5). Then we can solve the equilibrium futures price in its closed form: [10]

$$
\begin{aligned}
& F\left(W, r, V, M, N, \tau, t\right) \\
&= W(t)\, e^{\left[-\rho + \lambda_m\left(e^{-k} - 1\right)\right]\tau} \sum_{N=0}^{\infty} \left\{ \frac{e^{-\lambda_n \tau}\left(\lambda_n\, \tau\right)^N}{N!} \left[Q_1\, e^{\frac{1}{\lambda_n} Q_2 \times r(t) + \left(Q_2 - Q_3\right) \times V(t)} \right] \right\} \\
&= E_N\left[\Psi\left(W, r, V, M, N, \tau, t\right)\right],
\end{aligned} \tag{7}
$$

where

[9] Please refer to Appendix A for the proof.
[10] Please refer to Appendix B for the proof.

$$Q_1 = \left[\frac{2\,\kappa\, e^{\frac{c^2 \mu}{a-\kappa} N}}{(-a+\kappa)+e^{\kappa N}(a+\kappa)} \right]^{\frac{2ab}{c^2}} \times \left[\frac{2\,\nu\, e^{\frac{\sigma^2 h^2}{-f+\nu} N}}{(-f+\nu)+e^{\nu N}(f+\nu)} \right]^{\frac{2fg}{h^2}} \times e^{\frac{\lambda_m}{\lambda_n} \times k \times Q_2},$$

$$Q_2 = \frac{2\left(-1+e^{\kappa N}\right)}{a\left(-1+e^{\kappa N}\right)+\kappa\left(1+e^{\kappa N}\right)},$$

$$Q_3 = \frac{2\left(-1+e^{\nu N}\right)}{f\left(-1+e^{\nu N}\right)+\nu\left(1+e^{\nu N}\right)},$$

$$\kappa = \sqrt{a^2 - 2\,c^2\,\mu}\,,$$

$$\nu = \sqrt{f^2 + 2\,\sigma^2\,h^2}\,,$$

and

$$\Psi(W, r, V, M, N, \tau, t) = W(t)\, e^{\left[-\rho+\lambda_m\left(e^{-k}-1\right)\right]\tau}\, Q_1\, e^{\frac{1}{\lambda_n} Q_2 \times r(t) + \left(Q_2 - Q_3\right) \times V(t)}.$$

Note that M and N are two Poisson random variables in the time period $(T - t)$ originated from the Bernoulli variables dm and dn in an instantaneous time span dt, and $E_M(M) = \lambda_m \tau$ and $E_N(N) = \lambda_n \tau$. Ψ is the equilibrium futures price before expectation operation of the regular randomness of N. Equation (7) shows that the equilibrium futures price is an explicit function of variables W, r, V, τ, agent's intertemporal discount rate ρ, the expected impulse effect of percentage change associated with a jump event on the production activity k, and stochastic volatilities resulted from the regular and irregular information intensities controlled by instantaneous probabilities λ_n and λ_m.

Substituting $\tau = 0$ into Equation (7) verifies that the equilibrium stock index futures price satisfies the boundary condition $F(T) = W(T)$ at contract expiration date T. Namely, the futures price and spot price should be identical at the expiry time to satisfy the no-arbitrage condition. On the other hand, If M and N are non-stochastic with $\lambda_m = 0$ and $\lambda_n = 1$, then Equation (7) degenerates to the Hemler and Longstaff's (1991) solution. Moreover, when the state variables X and Y are non-stochastic except for M and N, the equilibrium interest rate r and market volatility V become constants and the solution degenerates into the cost of carry model.

III Properties of General-Equilibrium Pricing of Stock Index Futures

A. Comparative Static Results

Let us clarify the primary functional regularities and well-behaved conditions as follows. First, a greater mean-reverting speed parameter of a dynamics pulls outcomes back to its long-term average in a greater strength. Namely, the interest rate r and the regular market volatility V become more stable as a and f increase. Similarly, smaller diffusion terms in their dynamics create the same effect. Second, Q_1 in Equation (7) must be positive to ensure nonnegative futures price levels. Such a condition is accomplished by requiring that the economy is stable to some extent between one mean-reverting tendency and two fluctuating sources, i.e., the interest rate mean-reverting speed a, the interest rate volatility c^2, and the physical production growth parameter μ. Albeit Q_3 is positive because of the positive ν from its positive constituents, a^2 must be greater than or equal to $2c^2\mu$ in order to guarantee that both Q_2 and Q_1 are positive. Third, both probabilities λ_m and λ_n are nonnegative, and the sign of expected irregular impulse effect k depends on the property of rare event. For instance, k is positive for new major technology innovations and negative for catastrophes. Finally, various comparative static results via partial differential can be obtained from Equation (7):

$$\frac{\partial F}{\partial W}=\frac{F}{W}, \tag{8}$$

$$\frac{\partial F}{\partial r}=\frac{1}{\lambda_n}E_N\left(Q_2\times\Psi\right), \tag{9}$$

$$\frac{\partial F}{\partial \rho}=-\tau\times F, \tag{10}$$

$$\frac{\partial F}{\partial \tau}=E_N\left(\frac{N-\lambda_n\,\tau}{\tau}\times\Psi\right)+\left[-\rho+\lambda_m\left(e^{-k}-1\right)\right]F, \tag{11}$$

$$\frac{\partial F}{\partial \lambda_n}=E_N\left(\frac{N-\lambda_n\,\tau}{\lambda_n}\times\Psi\right)-\left(\frac{r+\lambda_m\,k}{\lambda_n}\right)\frac{\partial F}{\partial r}, \tag{12}$$

$$\frac{\partial F}{\partial \lambda_m}=\left(e^{-k}-1\right)\tau\,F+\frac{k}{\lambda_n}\times\frac{\partial F}{\partial r}, \tag{13}$$

$$\frac{\partial F}{\partial k}=-\lambda_m\,e^{-k}\,\tau\,F+\frac{\lambda_m}{\lambda_n}\times\frac{\partial F}{\partial r}, \tag{14}$$

and

$$\frac{\partial F}{\partial V}=E_N\left[\left(Q_2-Q_3\right)\Psi\right]=\lambda_n\frac{\partial F}{\partial r}-E_N\left(Q_3\,\Psi\right). \tag{15}$$

Just as in the cost of carry model, Equations (8), (9), and (10) show that F, the general-equilibrium price of index futures is positively correlated with the stock index W and interest rate r, and negatively correlated with the time preference parameter or dividend yield ρ. From Equation (11), however, the length of time to maturity τ is not necessarily positively correlated with the futures price because of the indeterminate combined effect of the time preference parameter ρ, regular and irregular information intensities λ_n and λ_m, and the expected impulse effect *k*. Similarly, partial effects resulted from the regular, irregular information arrival intensities, λ_n and λ_m, and the expected impulse effect k are complicated with relationships among themselves, the length of time to maturity τ, and the sensitivity of futures price to interest rate $\frac{\partial F}{\partial r}$ in Equations (12), (13), and (14).

On the other hand, the relationship between the stock index futures price and the regular volatility V depends on the relative size of Q_2 and Q_3 in Equation (15). Note that Q_2 and Q_3 are related to the growth and variability features respectively in the physical production process or Equation (3). The two variables affect equilibrium futures price in opposite directions since κ and ν are negatively proportional to Q_2 and Q_3 and thus the relative bounded degree of the growth feature to the variability feature affects the impact from regular volatility V.[11] Also note that the sensitivity of futures price to interest rate, which is governed by Q_2 plays an non-trivial role because futures price is affected by the impulse effect, regular, and irregular information intensities through their interactions with the equilibrium interest rate.

B. Simulation Results

Market participants may perceive the outlooks of economies including the growth and variability features, but they can not identify the underlying factors. Thus, we categorize economies by different underlying factors including the regular and irregular information intensities and mean-reverting speeds of state variables. Simulation results are provided to help clarifying relationships between key variables and explaining the cause of differential outcomes in economies with identical

[11] That is, a larger a (or κ) and a larger f (or ν) guarantee more converging strengths of X to b and Y to g, which resemble economic bounded degrees of the growth trend and variability feature in the production process dp.

outlooks.

Specifically, the stock index dynamics can be stated as in Equation (16).[12] For an illustration purpose, the perceived growth and variability features are defined and dichotomized by the levels of $\mu_W\left(X, n, t\right)$ and $\sigma_W\left(Y, n, t\right)$, and the regular and irregular information intensities are dichotomized by the levels of λ_n and λ_m. The cases $a > f$ and $a < f$ are taken into consideration as well, where a and f are the mean-reverting speed parameters of state variables X and Y in Equation (2). On controlling κ and ν or Q_2 and Q_3 in Equation (15), $a > f$ implies that the growth dynamic is more stable or more adhered to its long-term mean than the variability dynamics in the production process.

$$\frac{dW}{W} = \mu_W\left(X, n, t\right) dt + \sigma_W\left(Y, n, t\right) dz_{p(t)} + k\, dm\,. \tag{16}$$

Quantitative definitions of the key features are presented in Table 1. Other parameter settings for each economy are detailed in Table 2. The coding rule for each economy in Table 2 corresponds to the first four fundamental features defined in Table 1. For instance, the HLHL economy is with a "High" growth feature ($\mu_W = 0.4\%$), a "Low" variability feature ($\sigma_W = 0.1\%$), a "High" regular information intensity ($\lambda_n = 80\%$), and a "Low" occurrence probability of jump events ($\lambda_m = 0\%$). Note that some parameters are designated to have same growth and variability features perceived by agents for each economy. For example, both growth

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[12] Equation (16) is identical to Equation (3) when $\sigma_W\left(Y, n, t\right)$ is defined to be $\sqrt{V}\, dz_{p(n)}$ and when $\mu_W\left(X, n, t\right)$ is defined to be $\left[\mu\, X\, dn - \left(\lambda_m\, k + \rho\right) dt\right]$.

and variability features in the stock index dynamics of the HHHH and the HHLL economies are the same (μ_W^{HHHH} = μ_W^{HHLL} = 0.4% and σ_W^{HHHH} = σ_W^{HHLL} = 0.2%). However, given different levels of regular and irregular information intensities (λ_n^{HHHH} = 80% ≠ λ_n^{HHLL} = 40% and λ_m^{HHHH} = 1% ≠ λ_m^{HHLL} = 0%), the tallied or implied parameters may vary markedly, e.g. μ^{HHHH} = 0.0250% ≠ μ^{HHLL} = 0.1000%.

Moreover, Table 3 summarizes the signs of the partial derivatives of futures price by conducting a 100-period simulation with 10,000 paths for each economy. Note that the numbers in parentheses denote the turning points of length of time to maturity or nearness, at which the directions of relationships between futures price and other key variables are reversed for each case. For example, futures price is negatively related with regular information intensity when the contract is to be due in four periods in Economy LHHL given the $a > f$ case. However, the negative relationship changes to be positive when the nearness of the futures contract is greater or equal to five periods. Also note that the relationships between futures price and λ_n, λ_m, or k are more consistent in economies with same levels of regular and irregular information intensities (**HH, **HL, **LH, or **LL) and same relative level of mean-reverting speed ($a > f$ or $a < f$) than those are only in common in levels of growth and variability features (HH**, HL**, LH**, or LL**). Thus, categorization by the levels of regular-, irregular information intensity, and relative mean-reverting speed performs better in discerning economic phenomena than the categorization by the growth and variability features only.

Table 1 Quantitative Definition of the Primary Features

Primary Features	High	Low
Growth feature (μ_W)	0.40%	0.30%
Variability feature (σ_W)	0.20%	0.10%
Regular information intensity (λ_n)	80.00%	40.00%
Irregular information intensity (λ_m)	1.00%	0.00%
Mean-reverting speed of growth feature (a)	90.00%	60.00%
Mean-reverting speed of variability feature (f)	60.00%	90.00%

Table 2 Parameter Settings across Different Economies

Abbreviations for the Features of Economies	μ	σ	Abbreviations for the Features of Economies	μ	σ
Tallied parameters to keep the same growth and variability features					
HHHH	0.0250%	0.0632%	LHHH	0.0125%	0.0632%
HHLH	0.0500%	0.0632%	LHLH	0.0250%	0.0632%
HHHL	0.0500%	0.0632%	LHHL	0.0375%	0.0632%
HHLL	0.1000%	0.0632%	LHLL	0.0750%	0.0632%
HLHH	0.0250%	0.0316%	LLHH	0.0125%	0.0316%
HLLH	0.0500%	0.0316%	LLLH	0.0250%	0.0316%
HLHL	0.0500%	0.0316%	LLHL	0.0375%	0.0316%
HLLL	0.1000%	0.0316%	LLLL	0.0750%	0.0316%

Common parameter settings

b	c	g	h	τ	k	ρ	X_0	Y_0	W_0
10	1.50%	10	1.50%	100	-20%	0	10	10	1,000

Table 3 Simulated Signs of Partial Derivatives of Futures Price with Respect to the Exogenous Variables Corresponding to Various Features

Categorized by Regular and Irregular Information Intensities										
	τ	V	λn	λm	k	τ	V	λn	λm	k
	$a > f$					$a < f$				
HHHH	+	–	–	+	–	–	+	–	+	–
HLHH	+	–	–	+	–	–	+	–	+	–
LHHH	–	–	–	+	–	–	+	–	+	–
LLHH	–	–	–	+	–	–	+	–	+	–
HHHL	+	–	– (≤1) +	+	n.a.	–	+	–	+	n.a.
HLHL	+	–	– (≤1) +	+	n.a.	–	+	–	+	n.a.
LHHL	–	–	– (≤4) +	+	n.a.	–	+	–	+	n.a.
LLHL	–	–	– (≤3) +	+	n.a.	–	+	–	+	n.a.
HHLH	+	–	– (≤3) +	– (≤3) +	– (≤2) –	+	+	–	– (≤6) +	+ (≤5) –
HLLH	+	–	– (≤3) +	– (≤3) +	+ (≤2) –	+	+	–	– (≤6) +	+ (≤5) –
LHLH	+	–	–	– (≤3) +	+ (≤2) –	–	+	–	– (≤6) +	+ (≤5) –
LLLH	+	–	–	– (≤3) +	+ (≤2) –	–	+	–	– (≤6) +	+ (≤5) –
HHLL	+	–	+	– (≤3) +	n.a.	–	+	– (≤7) +	– (≤6) +	n.a.
HLLL	+	–	+	– (≤3) +	n.a.	+	+	– (≤7) +	– (≤6) +	n.a.
LHLL	+	–	+	– (≤3) +	n.a.	+	+	– (≤19) +	– (≤6) +	n.a.
LLLL	+	–	+	– (≤3) +	n.a.	–	+	– (≤19) +	– (≤6) +	n.a.

Note: "n.a." results from the "Low irregular information intensity" feature ($\lambda_m = 0\%$). The number in parentheses denotes the length of time to maturity, namely, the nearness of a futures contract, at which the sign of a partial derivatives happens to change.

The categorization by different features helps explain why different phenomena appear in economies with a similar outlook. A related issue is whether the contango or the backwardation pattern prevails.[13] According to the simulated relationship of futures prices F versus length of time to maturity τ in Table 3, even though economies are equipped with identical growth and variability features (HH**, HL**, LH**, or LL**), no clear-cut answer can be obtained without further discerning their relative mean-reverting speed of the growth and variability dynamics ($a > f$ or $a < f$) and the impacts of the regular and irregular information intensities (λ_n and λ_m).

Another related issue is how volatility affects futures price. Hemler and Longstaff (1991) find that market volatility affects futures prices in different directions across futures contracts with different lengths of maturity. While Chen, Cuny, and Haugen (1995) argue that due to increasing hedge demands and short positions for fear of the bear market, volatility should be negatively related to the futures price. According to our Table 3, the effect on futures price from the variability feature V primarily depends on the relative mean-reverting speed of the growth and variability dynamics, and remains monotonic given different fundamental features and lengths of time to maturity. Moreover, the result is robust with respect to various levels of a and f. If a is greater than f, then the growth dynamics is relatively stable and V and F are negatively correlated, which is consistent with Chen, et al. (1995). In contrast, if a is less than f, then the growth dynamics is relatively volatile and V and F are positively correlated. Since a more volatile underlying asset dynamics suggests a larger probability of extreme spot prices, we conjecture that in markets full of

[13] The phenomenon "contango" occurs when futures price is greater than spot price, and "backwardation" occurs when futures price is less than spot price, whereas "normal backwardation" occurs when futures price at current time is smaller than the "expected" spot price at expired date. The terminologies were defined by Keynes and used by Hicks (1946) latter. Because of the log utility and frictionless assumption in our model, the current time futures price equals the expected spot price at the expiry time. Thus, we focus on the contango and backwardation phenomena.

sentiments or mis-reaction on the growth perspective, the reported correlation between futures price and regular volatility are more likely to be positive.

IV Conclusion

This paper is among one of the studies motivated by the documented tail-fatness and leptokurtic observed in asset returns. The special features of this study are as follows. First, we take into account not only the regular but also the irregular stochastic volatility in explaining the futures price. Second, we apply the information-time setting to control the synchronic variation between the growth and variability features of an economy. Decomposing factors such as regular-, irregular information intensity, and mean-reverting speed further help to explain why correlation coefficients among key variables may differ in sign or magnitude even when the stock index dynamics of different economies looks alike. Unveiling these factors concealed in identical growth and variability features reconciles discrepancies about relationships among prices and volatility in futures and cash markets in literature.

Appendix A — Proof of Equation (6)

The dynamics of the representative agent's wealth W can be stated in details as:

$$
\begin{aligned}
dW &= w_p\, W \frac{dp}{p} + \sum_i w_{F^i}\, W \frac{dF^i}{F^i} + \left(1 - w_p - \sum_i w_{F^i}\right) W\, r\, dt - C\, dt \\
&= \left\{ \left[-w_p\, W\, \lambda_m\, k + \left(1 - w_p - \sum_i w_{F^i}\right) W\, r - C \right] \right. \\
&\qquad \left. + \left[\sum_i \frac{w_{F^i}}{F^i} W \left(F_t^i - F_W^i \left(W \lambda_m\, k + C \right) \right) \right] \Bigg|_{F_\bullet(m=0)} \right\} \times dt \\
&\quad + \left\{ w_p\, W\, \mu\, X + \left[\sum_i \frac{w_{F^i}}{F^i} W \left(F_W^i\, W\, \mu\, X + F_X^i\, a\left(b - X\right) + F_Y^i\, f\left(g - Y\right) \right. \right. \right. \\
&\qquad \left. \left. \left. + \frac{1}{2} F_{WW}^i\, W^2\, \sigma^2\, Y + \frac{1}{2} F_{XX}^i\, c^2\, X + \frac{1}{2} F_{YY}^i\, h^2\, Y \right) \right] \Bigg|_{F_\bullet(m=0)} \right\} \times dn \\
&\quad + \left[w_p\, W\, \sigma \sqrt{Y} + \left(\sum_i \frac{w_{F^i}}{F^i} W\, F_W^i\, W\, \sigma \sqrt{Y} \right) \Bigg|_{F_\bullet(m=0)} \right] dz_{p(n)} \\
&\quad + \left(\sum_i \frac{w_{F^i}}{F^i} W\, F_X^i\, c \sqrt{X} \right) \Bigg|_{F_\bullet(m=0)} dz_{X(n)} + \left(\sum_i \frac{w_{F^i}}{F^i} W\, F_Y^i\, h \sqrt{Y} \right) \Bigg|_{F_\bullet(m=0)} dz_{Y(n)} \\
&\quad + \left\{ w_p\, W\, k + \sum_i \frac{w_{F^i}}{F^i} W \left[F^i\left(W, X, Y, m = 1, t\right) - F^i\left(W, X, Y, m = 0, t\right) \right] \right\} dm .
\end{aligned}
$$

Note that the occurrences of consumption and interest income from the riskless position are based on the calendar-time instead of the information-time spans. That is, the instantaneous consumption and interest occur whether or not an information arrival does happen in that time span. On the other hand, we can obtain the dynamics of the agent's expected lifetime value function $E\left(J\right)$ by Itô lemma:

$$
\begin{aligned}
\Lambda \times dt \equiv \max_{w_p \; w_F} E_{W,\,X,\,Y,\,m,\,n,\,t} & \left[e^{-\rho t} \ln\big(C(t)\big)\, dt + dJ\big(W, X, Y, m, t\big) \right] \\
= e^{-\rho t} \ln\big(C(t)\big)\, dt + E_t \Bigg[& \; J_t \, dt + J_W \, dW + J_X \, dX + J_Y \, dY \\
& + \frac{1}{2} J_{WW} \left(dW\right)^2 + \frac{1}{2} J_{XX} \left(dX\right)^2 + \frac{1}{2} J_{YY} \left(dY\right)^2 \\
& + J_{WX} \left(dW\right)\left(dX\right) + J_{WY} \left(dW\right)\left(dY\right) + J_{XY} \left(dX\right)\left(dY\right) \Bigg] \Bigg|_{J_{\bullet}(m=0)} \\
+ \lambda_m \Big[J\big(W, X, Y, m=1, t\big) - J\big(W, X, Y, m=0, t\big) \Big] \, dt \, . &
\end{aligned}
$$

Applying both first-order conditions ($\dfrac{\partial \Lambda}{\partial w_p} = \dfrac{\partial \Lambda}{\partial w_{F^i}} = 0$) and market clearing conditions ($w_p = 1$ and $w_{F^i} = 0, \forall i$), we can obtain Equations (3) and (4). Equation (6) can also be obtained after tedious algebraic manipulations.

Appendix B — Proof of Equation (7)

We assume that the analytic form of futures pricing is multiplicatively separable and can be stated as:

$$
F\big(W, X, Y, m, \tau, t\big) = W(\tau)\, A(\tau)\, e^{B(\tau)\, X + C(\tau)\, Y}, \qquad \tau \equiv T - t \, .
$$

It is straightforward to show that the trail solution satisfies Equation (6). Substituting the trail solution into the partial differential equation, we obtain the three following sub-equations:

$$
\frac{A_\tau}{A} - \big(\lambda_n \, a \, b\big) B - \big(\lambda_n \, f \, g\big) C + \lambda_m \, k = 0 \, , \qquad \text{(B-1)}
$$

$$B_\tau + \left(\lambda_n\ a\right)B - \left(\frac{1}{2}\lambda_n\ c^2\right)B^2 - \lambda_n\ \mu = 0\,, \qquad \text{(B-2)}$$

and

$$C_\tau + \left(\lambda_n\ f\right)C - \left(\frac{1}{2}\lambda_n\ h^2\right)C^2 + \lambda_n\ \sigma^2 = 0\,. \quad \text{(B-3)}$$

Note that (B-2) and (B-3) are in the form of Ricatti equation and that *A* and *B* may be solved in their closed forms:

$$B(\tau) = \frac{2\,\mu\left[-1+e^{\kappa N}\right]}{a\left[-1+e^{\kappa N}\right]+\kappa\left[1+e^{\kappa N}\right]}, \qquad \kappa = \sqrt{a^2 - 2\,c^2\ \mu}\,,$$

and

$$C(\tau) = \frac{2\,\sigma^2\left[1-e^{\nu N}\right]}{f\left[-1+e^{\nu N}\right]+\nu\left[1+e^{\nu N}\right]}, \qquad \nu = \sqrt{f^2 + 2\,\sigma^2\ h^2}\ .$$

Then (B-1) is readily to be solved as well:

$$A(\tau) = e^{(-\rho\,\tau - k\,M)} \times \left\{ e^{\frac{2\,a\,b\,\mu}{a-\kappa}N} \left[\frac{2\,\kappa}{(-a+\kappa)+e^{\kappa N}(a+\kappa)}\right]^{\frac{4\,a\,b\,\mu}{a^2-\kappa^2}} \right\}$$
$$\times \left\{ e^{\frac{2\,f\,g\,\sigma^2}{-f+\nu}N} \left[\frac{2\,\nu}{(-f+\nu)+e^{\nu N}(f+\nu)}\right]^{\frac{4\,f\,g\,\sigma^2}{-f^2+\nu^2}} \right\}.$$

We can then apply Equations (4) and (5) to transform the solution $W(\tau)\ A(\tau)\ e^{B(\tau)\,X + C(\tau)\,Y}$ into Equation (7). Substituting $\tau = 0$ into Equation (7) satisfies the boundary condition $F(T) = W(T)$ at contract expiration date *T*. On the other hand, If *M* and *N* are non-stochastic with $\lambda_m = 0$ and $\lambda_n = 1$, then Equation (7) degenerates to the setting in Hemler and Longstaff (1991). Moreover, when not only *M* and *N* but also state variables *X* and *Y* are non-stochastic, the equilibrium interest rate *r* and market volatility *V* become constants and the setting degenerates into the cost of carry model.

References

Amin, K. I., and V. Ng, 1993, Option valuation with systematic stochastic volatility, *Journal of Finance* 48, 881-910.

Bakshi, G., Cao, C., and Z. Chen, 1997, Empirical performance of alternative option pricing models, *Journal of Finance* 52, 2003-2049.

Bates, D., 1991, The crash of 87: Was it expected? The evidence from options markets, *Journal of Finance* 46, 1009-1004.

Bates, D., 1996, Jumps and stochastic volatility: exchange rate processes implicit in deutsche mark options, *Review of Financial Studies* 9, 69-107.

Bates, D., 2000, Post-'87 crash fears in S&P 500 futures options, *Journal of Econometrics* 94, 181-238.

Black, F. and M. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 81, 637-659.

Blattberg, R. C. and N. J. Gonedes, 1974, A comparison of the stable and student distributions as statistical model for stock prices, *Journal of Business* 47, 244-280.

Chang, C. W., Jack, S. K. Chang, and K. G. Lim, 1998, Information-time option pricing: theory and empirical evidence, *Journal of Financial Economics* 48, 211-242.

Chen, N., Cuny, C. J., and R. A. Haugen, 1995, Stock volatility and the levels of the basis and open interest in futures contracts, *Journal of Finance* 50, 281-300.

Clark, P. K., 1973, A subordinated stochastic process model with finite variance for speculative prices, *Econometrica* 41, 135-155.

Cox, J. C. and S. A. Ross, 1976, The valuation of options for alternative stochastic process, *Journal of Financial Economics* 3, 145-166.

Cox, J. C., Ingersoll, J. E., and S. A. Ross, 1981, The relation between forward prices and futures prices, *Journal of Financial Economics* 9, 321-346.

Cox, J. C., Ingersoll, J. E., and S. A. Ross, 1985, An intertemporal general equilibrium model of asset prices, *Econometrica* 53, 363-384.

Chesney, M. and L. Scott, 1989, Pricing European currency options: A comparison of the modified Black–Scholes model and a random variance model, *Journal of Financial and Quantitative Analysis* 24, 267-284.

Duffie, D., Pan, J. and K. Singleton, 2000, Transform analysis and asset pricing for affine jump diffusion, *Econometrica* 68, 1343-1376.

Eraker, B., 2004, Do stock prices and volatility jump? Reconciling evidence from spot and option prices, *Journal of Finance* 59, 1367-1403.

Eraker, B., Johannes, M. S., and N. G.. Polson, 2003, The impact of jumps in returns and volatility, *Journal of Finance* 53, 1269–1300.

Fama, E., 1965, The Behavior of Stock Prices, *Journal of Business* 47, 244-280.

Hemler, M. L. and F. A. Longstaff (1991), General equilibrium stock index futures prices: Theory and empirical evidence, *Journal of Financial and Quantitative Analysis* 26, 287-308.

Heston, S., 1993, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies* 6, 327-343.

Hicks, J. R., 1946, Value and capital, *Clarendon Press*, Oxford, second edition.

Hilliard, J. E. and J. Reis, 1998, Valuation of commodity futures and options under stochastic convenience yields, interest rates, and jump diffusions in the spot, *Journal of Financial and Quantitative Analysis* 33, 61-86.

Hull, J. and A. White, 1987, The pricing of options on assets with stochastic volatilities, *Journal of Finance* 42, 281-300.

Jarrow, R. and G. Oldfield, 1981, Forward contracts and futures contracts, *Journal of Financial Economics* 9, 373-382.

Jorion, P., 1988, On jump processes in the foreign exchange and stock markets, *Review of Financial Studies* 1, 427-445.

Kon, S. J., 1984, Models of stock returns: A Comparison, *Journal of Finance* 39, 147-166.

Liu, J., Longstaff, F. A., and J. Pan 2003, Dynamic asset allocation with event risk, *Journal of Finance* 58, 231-259.

Melino, A. and S. M. Thornbull, 1990, Pricing foreign currency options with systematic stochastic volatility, *Journal of Econometrics* 45, 239-265.

Merton, R. C., 1976, Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics* 3, 125-144.

Naik, V. and M. Lee, 1990, General equilibrium pricing of options on the market portfolio with discontinuous returns, *Review of Financial Studies* 3, 493-521.

Pan, J., 2002, The jump-risk premia implicit in options: Evidence from an integrated time-series study, *Journal of Financial Economics* 63, 3-50.

Paretz, P., 1972, The distributions of share price changes, *Journal of Business* 45, 49-55.

Richard, S. F. and M. Sundaresan, 1981, A continuous time equilibrium model of forward prices and futures prices in a multigood economy, *Journal of Financial Economics* 9, 347-371.

Scott, L. O., 1997, Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: Applications of Fourier inversion methods, *Mathematical Finance* 7, 413-428.

Shiller, R. J., 2002, Bubbles, human judgment, and expert opinion, *Financial Analysts Journal* 58, 18-26.

Stein, E. M., and J. C. Stein, 1991, Stock price distributions with stochastic volatility: an analytic approach, *Review of Financial Studies* 4, 727-752.

Wiggins, J. B., 1987, Option values under stochastic volatility: Theory and empirical estimates, *Journal of Financial Economics* 19, 351-372.