Optimal Dynamic Option-Based Portfolio Insurance Strategies with Stochastic Volatility

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Abstract

The intertemporal investment-consumption technique is applied to investigate the optimal consumption and dynamic option-based portfolio insurance strategy when there is predictable variation in return volatility. An optimal dynamic option-based portfolio insurance strategy is shown to be separable into a myopic component and an intertemporal hedging component. The intertemporal hedging demand is further separated into three effects. The correlation effect results in a conservative investor having a negative position on the intertemporal hedging demand of the option-based portfolio insurance strategy. However, there are two other positive effects in the intertemporal hedging component for the option-based portfolio insurance: the delta effect and the vega effect. Incorporating options' considerations in portfolio decisions to create a dynamic option-based portfolio insurance strategy improves the hedging ability in the intertemporal hedging component, especially in down markets.

Keywords: Portfolio insurance strategies; Stochastic volatility; Intertemporal model

JEL classification: G11, C61, D90

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1. INTRODUCTION

In a world of uncertainty, steps are taken to offset or at least reduce the chance of loss or failure. In the modern financial market, a wide range of portfolio insurance strategies is available to investors or fund managers for this aim. Portfolio insurance strategies are appropriate for investors who need to limit downside risk and desire to participate in upside potential. Various portfolio insurance models are given in both the academic and professional literature. Among them, the two main and most popular strategies implementing portfolio insurance are the Option-Based Portfolio Insurance (OBPI) and the Cushion method (also known as the Constant Proportion Portfolio Insurance, or CPPI, method). Both methods are designed to guarantee that the portfolio's current value dominates the discounted value of a pre-specified floor.

The CPPI was introduced by Perold (1986) and Perlod and Sharpe (1988) and has been further developed and explored by Black and Jones (1987), Black and Rouhani (1989), and Black and Perold (1992). The CPPI strategy basically buys shares as they rise and sells shares as they fall. This method is a dynamic strategy in which the investor starts choosing the lowest acceptable value of the portfolio (usually called the floor, which is the function of the investor's preference and risk tolerance). If we think of the difference between the assets and floor as a "cushion", then the investor shall maintain the portfolio's risk exposure at a constant multiple of the cushion, i.e. the excess of wealth over the floor (Black and Rouhani, 1989 and Black and Perold, 1992).

Another strategy is the OBPI, pioneered by Leland and Rubinstein (1976), consisting basically of simultaneously buying the risky asset or stock and a traded or synthetic put written on it. For instance, a portfolio insurer might buy a put option on a stock index, giving him the right to sell the index at a predetermined price. If the index falls below that price in the future, then the insurer could exercise or sell the put with a profit, which would compensate for the reduced value of the insurer holding stock. On the other hand, if the stock rises, then all the insurer loses is the premium for the put, while enjoying the rise in the value of the stock held.

This paper mainly aims to provide an investment strategy for one special kind of investor, portfolio insurers, such as some institutional investors. This kind of investment strategy is called portfolio insurance strategy. This kind of investor will and should build together options and stocks in a single portfolio, in which the single portfolio is just called portfolio insurance.

Leland (1980) concludes that two kinds of investors wish to obtain portfolio insurance. The first kind of investor has average expectations, but the risk tolerance increases with wealth more rapidly than average. The second kind of investor has average risk tolerance, but the expectations of returns are more optimistic than average. Purchasing portfolio insurance is equivalent to holding a reference portfolio and buying a put option on the portfolio. Therefore, the definition of portfolio insurance is: a strategy of hedging a stock portfolio against market risk by selling stock index futures short or buying stock index put options.

Portfolio insurance is equivalent to a securities position comprised of an underlying portfolio plus an insurance policy that guarantees the portfolio against loss through a specified policy expiration date (Rubinstein, 1985). Leland and Rubinstein (1976) introduced the Option Based Portfolio Insurance (OBPI), which is one of the more popular and widely-used strategies of portfolio insurance by portfolio insurers such as mutual funds or pension funds. It consists basically of "buying simultaneously a stock (generally a financial index) and a put written on it". The value of this portfolio at maturity is always greater than the strike of the put, whatever the market fluctuations are. The purpose of portfolio insurance is to insure a minimum value for a stock portfolio in a falling market, while also allowing for participation in a rising market. For instance, a portfolio insurer might simultaneously buy a share of a stock index portfolio, like S&P500, and a put option written on the S&P 500, giving him the right to sell the index at a predetermined level. If the index falls below that level, then the insurer exercises or sells the put. The profit on the put offsets the decline in the value of the stocks the insurer holds. If stocks in the index rise, the insurer loses what he paid for the put.

Therefore, this kind of investor, a portfolio insurer, must build together options and stocks in a single portfolio. From this basic issue, the paper further discusses and explores the optimal dynamic option-based portfolio insurance strategy with stochastic volatility. In this paper we apply the intertemporal investment-consumption technique to discuss and explore the optimal dynamic option-based portfolio insurance strategy with stochastic volatility. More specifically, what we try to do in this paper that differs from the OBPI literature consists of the following parts.

First, the traditional option-based portfolio insurance strategy has both of the following goals: to protect the portfolio value "at maturity" and to take advantage of rises in the underlying "tactical allocation". As analyzed in this paper, we allow our model to get the optimal dynamic option-based portfolio insurance strategy at any time instead of just to guarantee a fixed amount only at the terminal date. This paper sets up a model in which a long-term investor chooses an optional dynamic option-based portfolio insurance strategy by maximizing a utility function defined over intermediate consumption rather than terminal wealth. The abstraction from the choice of consumption over time implies that investors value only wealth at a single terminal date, i.e. no consumption takes place before the terminal date, and all portfolio returns are reinvested until that date (Campbell and Viceira, 1999). In addition, the assumption that investors derive utility only from terminal wealth and not from intermediate consumption will simplify the analysis by avoiding an additional source of non-linearity in the differential equation. However, many long-term investors desire to seek a stable consumption path over a long horizon. This simplification makes it hard to apply the model to the realistic problem facing an investor saving for the future. Very often, intermediate consumption can be used as an indicator of marginal utility, especially in the asset pricing related literature (Campbell and Viceira, 1999).

We know that the second goal of the option-based portfolio insurance is to take advantage of rises in the underlying "tactical allocation". The definition of tactical asset allocation strategies is essentially single-period or myopic asset allocation strategies which assume that the decision maker has a mean-variance criterion defined over the one-period rate of return on the portfolio (Brennan, Schwartz and Lagnado, 1997). However, a multi-period dynamic asset allocation is more reasonable and realistic than tactical asset allocation strategies for long-horizon investors. Merton (1971, 1973) shows that when investment opportunities are time-varying, dynamic hedging is necessary for forward-looking investors. Multi-period or long-horizon investors are concerned not only with expected returns and risk today, but with ways in which expected returns and risk may change over time. Dynamic asset allocation strategies for multi-period or long-horizon investors differ from those of single-period investors, because the former demand risky assets not only for their risk premia, but also for their hedging ability on consumption against adverse changes in future investment opportunities. Merton is the first to consider the effect of a stochastic investment opportunity set in the analysis of optimal asset allocation strategies for long-horizon investors. Merton (1969, 1971, 1973) shows that if investment opportunities are varying overtime, then long-horizon investors generally care about shocks to investment opportunities and not just about wealth itself. They may seek to hedge their exposures to wealth shocks, and this creates intertemporal hedging demand for financial assets (Campbell, 2000).

In this paper we choose the intertemporal model to discuss the more general hedging needs of the long-term investor. The application of the intertemporal investment model (Merton, 1971) to the problem of portfolio insurance is the “strategic asset allocation” instead of “tactical asset allocation”. The strategic asset allocation also allows investors to obtain growth, while limiting the chances of huge losses. While there is an abundant amount of literature exploring intertemporal strategic asset allocation, there is not much research exploring the intertemporal option-based portfolio insurance strategy. Because of time variation in investment opportunities, multi-period investors have an extra demand for financial assets (stocks or bonds) that reflect intertemporal hedging. That intertemporal hedging component of the optional portfolio strategies for multi-period investors is the major difference from those of single period investors. We also know that the intertemporal hedging component of the risky financial assets depends on the instantaneous correlation between the risky asset returns and state variable, implying that if shocks to expected returns are instantaneously perfectly correlated with shocks to realized returns or, equivalently, that markets are complete, then the intertemporal hedging component of the risky asset can be provided perfectly, or there is full hedging ability for multi-period investors (Schroder and Skiadas, 1999, Wachter, 2002 and Brennan and Xia, 2002).

If we allow for imperfect instantaneous correlation between risky asset returns and the state variable in our model, then the intertemporal hedging component of the risky asset can only provide partial hedging ability for multi-period investors when facing the time-varying investment opportunity set. In addition, the best portfolio insurance strategies should be found by solving for the intertemporal investment-consumption rules that maximize expected utility (Black and Perold, 1992). At this time, we can introduce (enough) non-redundant derivatives in the incomplete financial market to create the option-based portfolio insurance strategies as shown in our model. The intertemporal option-based portfolio insurance strategy can supplement the deficient hedging ability of the intertemporal hedging compound of the risky stock, because of the non-linear nature of derivatives.

If asset returns or volatility are time-varying, then this again implies that investment opportunities are time-varying, too, and an intertemporal model is needed to find the optimal asset allocation (Campbell, 2000). Intertemporal hedging and asset allocation are quantitatively important in light of the observed predictable variation in volatility as seen in Lynch and Balduzzi (2000), Barberis (2000), Brandt (1999), Brennan, Schwartz and Lagnado (1997), Campbell and Viceira (1999, 2001), Campbell, Chan and Viceira (2003) and Chacko and Viceira (2005). In this paper we incorporate, in addition to the usual diffusive price shock, stochastic volatilities that are important in characterizing the stock market to our option-based portfolio insurance strategy model. We also allow for imperfect instantaneous correlation between the price and volatility shocks－a feature that is important in the data. The option-based portfolio insurance strategy model is necessary and important to complete the market with respect to volatility risk.

In this generalized intertemporal model under the stochastic environment, Merton’s approach (1971, 1973) could not be used to derive a closed-form solution by solving a non-linear differential equation on the intertemporal hedging portfolio. Recently, some studies in the literature have begun to work on it, such as the approximate analytical solutions developed by Campbell and Viceira (2001), Kogan and Uppal (2001), and Chacko and Viceira (2005). These solutions are based on perturbations of known exact solutions. They offer analytical insights into investor behavior in models that fall outside the still limited class that can be solved exactly (Campbell, 2000). In this paper we use perturbation methods to get linear approximate solutions. We mainly derive the explicit solution on a log-linear expansion of the consumption-wealth ratio around its unconditional mean provided by Campbell (1993), Campbell and Viceira (1999, 2001 and 2002) and Chacko and Viceira (2005).

This paper is organized as follows. Section 2 describes the model used and environment assumed in this paper. Section 3 develops the model of optimal dynamic option-based portfolio insurance strategies with stochastic volatility. Section 4 provides analyses of the optimal dynamic option-based portfolio insurance strategy. Finally, conclusions are given in Section 5.

2. THE MODEL

2.1 Investment opportunity set

This paper assumes that the portfolio insurer invests wealth in traded assets only. There are two prime assets available for trading in the economy. One of the assets is a riskless money market fund, denoted by B_t with a constant interest rate of r. Its instantaneous return is:

$$\frac{dB_t}{B_t} = rdt \ . \tag{1}$$

The short rate is assumed to be constant in order to focus on the stochastic volatility of the risky asset. The second prime asset is a risky stock that represents the aggregate equity market. Here, S_t denotes the price of the risky financial asset at time t, and its instantaneous total return dynamics is given by:

$$\frac{dS_t}{S_t} = \mu\, dt + \sqrt{V_t}\, dZ_S\,, \tag{2}$$

where μ is the instantaneous expected rate of return on the risky stock, and $\sqrt{V_t}$ is the time-varying instantaneous standard deviation of the return on the risky asset. We denote stochastic variables with a subscript "t" and let the conditional variance of the risky stock vary stochastically over time.

From the following setting, the investment opportunity is time-varying. We assume that the instantaneous variance process is:

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}\, dZ_v\,, \tag{3}$$

where the parameter $\theta > 0$, which describes the long-term mean of the variance, and $\kappa \in (0,1)$ is the reversion parameter of the instantaneous variance process - i.e. this parameter describes the degree of mean reversion. Here, dZ_S and dZ_v are two Wiener processes with constant correlation ρ. We assume that the stock returns are correlated with changes in volatility with instantaneous correlation ρ, which may be assumed to be negative to capture the leverage effect or the asymmetric effect (Glosten et al., 1993). The negative correlation assumption with the mean-reversion on stock returns volatility can capture two of the most important features discussed in the empirical literature on the equity market.

In the traditional theory of derivative pricing (Black and Scholes, 1973 and Merton, 1973), derivative assets like options are viewed as redundant securities, for which the payoffs can be replicated by portfolios of primary assets. Thus, the market is generally assumed to be complete without the options. In this paper we introduce derivatives that allow the investor to include it in her dynamic asset allocation strategies to create option-based portfolio insurance. If only a risky stock and a riskless bond are available for trading, then the financial market is incomplete. This is from our setting that stock returns are not instantaneously perfectly correlated with their time-varying volatility. This paper set the derivatives written on the stock as a non-redundant asset. For our setting, the derivatives can provide differential exposure to the imperfect instantaneous correlation between volatility and stock returns, and they can make the market complete.

Following from Liu and Pan (2003), this paper provides a specific pricing kernel to price all of the risk factors in this economy and consequently the put options. The particular specification of the derivatives that complete the market is linked uniquely to a pricing kernel $\{\pi_t, 0 \le \mathrm{t} \le T\}$ such that $P_t = \frac{1}{\pi_t} E_t[\pi_\tau p(S_\tau, V_\tau)]$, for any $t \le \tau$; where τ is the time to expiration for the derivative security. In accordance with Liu and Pan (2003) we start with the following parametric pricing kernel, $d\pi_t = -\pi_t \left[rdt + (\mu - r)\sqrt{V_t} dZ_s + \lambda \sqrt{V_t} dZ_v \right]$, where $\pi_0 = 1$ and the constant coefficient $(\mu - r)$ and λ control the premium for the diffusive price risk and the stochastic volatility risk.

Following from Sircar and Papanicolaou (1999), Liu and Pan (2003), and from this pricing kernel, and under the above setting, the non-redundant put option ($P_t = p(S_t, V_t)$) which is the function (p) on the prices of the stock (S_t) and on the volatility of stock returns (V_t) at time t will have the following parametric specification of the price dynamics for the put options:

$$dP_t = [(\mu - r)S_t p_s + \lambda \sigma p_v + rP_t]\, dt + \sqrt{\mathrm{V}_t} S_t p_s dZ_s + \sigma \sqrt{V_t}\, p_v dZ_v \, . \tag{4}$$

where λ determines the stochastic volatility risk premium, and $p_s < 0$ and $p_v > 0$ are measures of the put option's price sensitivity to small changes in the underlying stock price and volatility, respectively.

The option-based portfolio insurance method consists basically of purchasing simultaneously q_t shares of the risky stock (S_t) and q_t shares of put options written on the stock with the non-linear payoff structure $p(S_\tau, V_\tau) = (K - S_\tau)^+$ for some strike price $K > 0$ at $t < \tau$. Thus, the dynamics of the portfolio value under option-based portfolio insurance (F_t) would be:

$$dF_t = [\mu S_t + (\mu - r) S_t p_s + \lambda \sigma p_v + rP_t]dt + [\sqrt{V_t} S_t + \sqrt{V_t} S_t p_s]dZ_s + \sigma \sqrt{V_t} p_v dZ_v . \qquad (5)$$

2.2 Preferences

We assume that the investor's preference is recursive and of the form described by Duffie and Epstein (1992). Recursive utility is a generalization of the standard and time-separable power utility function that separates the elasticity of intertemporal substitution of consumption from the relative risk aversion (Chacko and Viceira, 2005). This means that the power utility is just a special case of the recursive utility function when the elasticity of the intertemporal substitution is just the inverse of the relative risk aversion coefficient.

$$J = E_t[\int_t^{\infty} f(C_\tau, J_\tau)\, d\tau], \qquad (6)$$

where $f(C_\tau, J_\tau)$ is a normalized aggregator of investor's current consumption (C_τ) and utility has the following form:

$$f(C,J) = \beta(1 - \frac{1}{\varphi})^{-1}(1-\gamma)J\left[(\frac{C}{((1-\gamma)J)^{\frac{1}{1-\gamma}}})^{1-\frac{1}{\varphi}} - 1 \right], \qquad (7)$$

where γ is the coefficient of relative risk aversion, β is the rate of time preference, and φ is the elasticity of intertemporal substitution - they are all larger than zero.

The investor's objective is to maximize her expected lifetime utility by choosing consumption and the proportions of her wealth to invest in the option-based portfolio insurance subject to the following intertemporal budget constraint:

$$dW_t = \left[n_t \left(\mu \frac{S_t}{F_t} + (\mu - r) \frac{S_t}{F_t} p_s + \lambda \sigma \frac{p_v}{F_t} + \frac{rP_t}{F_t} - r \right) W_t + rW_t - C_t \right] dt$$

$$+ n_t \left(\sqrt{V_t} \frac{S_t}{F_t} + \sqrt{V_t} \frac{S_t}{F_t} p_s \right) dZ_s W_t + n_t \left(\sigma \sqrt{V_t} \frac{p_v}{F_t} \right) dZ_v W_t , \qquad (8)$$

where W_t represents the investor's total wealth, n_t are the fractions of the investor's financial wealth allocated to the option-based portfolio insurance at time t, and C_t represents the investor's instantaneous consumption.

3. OPTIMAL CONSUMPTION POLICY AND DYNAMIC OPTION-BASED PORTFOLIO INSURANCE STRATEGIES

The main objective of this paper is to explore the optimal option-based portfolio insurance strategies. Instead of a single period result, we also want to explore the optimal intertemporal consumption with a stochastic investment opportunity set induced by the stochastic volatility.

3.1 A special case with unit elasticity of intertemporal substitution of consumption

The value function of the problem (J) is to maximize the investor's expected lifetime utility. The principle of optimality leads to the following Bellman equation for the utility function. Under the above setting, the Bellman equation satisfies:

$$0=\sup_{n,C}\left\{f(C_\tau,J_\tau)+J_W\left[n_t\left(\mu\frac{S_t}{F_t}+(\mu-r)\frac{S_t}{F_t}p_s+\lambda\sigma\frac{p_v}{F_t}+\frac{rP_t}{F_t}-r\right)W_t+rW_t-C_t\right]+J_V\left[\kappa(\theta-V_t)\right]\right.$$

$$+\frac{1}{2}J_{WW}\left[n_t^2\left(\frac{S_t}{F_t}+\frac{S_t}{F_t}p_s\right)^2V_t+n_t^2\left(\sigma\frac{p_v}{F_t}\right)^2V_t+2n_t^2V_t\left(\frac{S_t}{F_t}+\frac{S_t}{F_t}p_s\right)\left(\sigma\frac{p_v}{F_t}\right)\rho\right]W_t^2$$

$$\left.+\frac{1}{2}J_{VV}\sigma^2V_t+J_{WV}W_t\left[n_t\left(\frac{S_t}{F_t}+\frac{S_t}{F_t}p_s\right)V_t\sigma\rho+n_t\left(\frac{p_v}{F_t}\right)\sigma^2V_t\right]\right\}, \tag{9}$$

where J_W and J_V. denote the derivatives of J with respect to wealth W and stochastic volatility V_t. We will use the similar notation for higher derivatives as well. We also note that ρ is the instantaneous correlation between the unexpected return on the stock and its stochastic volatility.

The first-order conditions for the equation are:

$$C_t=J_W^{-\varphi}J^{\frac{1-\varphi\gamma}{1-\gamma}}\beta^{\varphi}(1-\gamma)^{\frac{1-\varphi\gamma}{1-\gamma}}, \tag{10}$$

$$n_t=-\frac{J_W}{J_{WW}W_t}\frac{[(\mu-r)S_t(1+p_s)+\lambda\sigma p_v]F_t}{[S_t^2+S_t^2p_s^2+2S_t^2p_s+\sigma^2p_v^2+2\rho(\sigma S_tp_v+p_s\sigma S_tp_v)]V_t}$$

$$-\frac{J_{WV}}{J_{WW}W_t}\frac{\sigma\left(\rho S_t+\rho S_tp_s+\sigma p_v\right)F_t}{S_t^2+S_t^2p_s^2+2S_t^2p_s+\sigma^2p_v^2+2\rho(\sigma S_tp_v+p_s\sigma S_tp_v)}. \tag{11}$$

The optimal dynamic option-based portfolio insurance has two major components. In the option-based portfolio insurance, its first term is the mean-variance portfolio weight. This is for an investor who only invests in a single period horizon or under a constant investment opportunity set, the myopic demand. The second term of the optimal dynamic option-based portfolio insurance is the intertemporal hedging demand that characterizes demand arising from the desire to hedge against changes in the investment opportunity set induced by the stochastic volatility. This term is determined by the instantaneous rate of changes in relation to the value function. Aside from the partial hedging provided by the stock in the intertemporal hedging component, put options also offer additional hedging ability, allowing the investor to insure against changes in the stochastic volatility and investment opportunity set.

We will discuss this in more detail later, because the first-order conditions for our problem are not explicit solutions unless we know the complicated indirect utility function. Substituting the first-order solutions back into the Bellman equation, we get:

$$
\begin{aligned}
0 = {} & f(C(J),J) - J_W C(J) + J_W r W_t + J_V\left[\kappa(\theta - V_t)\right] + \frac{1}{2} J_{VV}\sigma^2 V_t \\
& - \frac{1}{2}\frac{(J_W)^2}{J_{WW}} \frac{\left((\mu - r)S_t(1+p_s) + \lambda\sigma p_v\right)^2}{[(S_t)^2 + (S_t)^2 p_s^2 + 2(S_t)^2 p_s + \sigma^2 (p_v)^2 + 2\rho(\sigma S_t p_v + p_s \sigma S_t p_v)]V_t} \\
& - \frac{1}{2}\frac{(J_{WV})^2}{J_{WW}} \frac{\sigma^2\left(\rho S_t + \rho S_t p_s + \sigma p_v\right)^2 V_t}{(S_t)^2 + (S_t)^2 p_s^2 + 2(S_t)^2 p_s + \sigma^2 (p_v)^2 + 2\rho(\sigma S_t p_v + p_s \sigma S_t p_v)} \\
& - \frac{J_{WV} J_W}{J_{WW}} \frac{\left((\mu - r)S_t(1+p_s) + \lambda\sigma p_v\right)\sigma\left(\rho S_t + \rho S_t p_s + \sigma p_v\right)}{(S_t)^2 + (S_t)^2 p_s^2 + 2(S_t)^2 p_s + \sigma^2 (p_v)^2 + 2\rho(\sigma S_t p_v + p_s \sigma S_t p_v)}. \qquad (12)
\end{aligned}
$$

We conjecture that there exists a solution of the functional form $J(W_t, V_t) = I(V_t)\frac{W_t^{1-\gamma}}{1-\gamma}$ when $\varphi = 1$, and after substituting it into equation (12), the ordinary differential equation will have a solution of the form $I = \exp(Q_0 + Q_1 V_t + Q_2 \log V_t)$. Rearranging that equation, we have three equations for Q_2, Q_1, and Q_0 after collecting terms in $\frac{1}{V_t}$, V_t and 1. We provide the full details in Appendix A.

We are now able to obtain the indirect utility function and the optimal consumption rule and dynamic option-based portfolio insurance strategy when $\varphi = 1$. The indirect utility function is:

$$J(W_t, V_t) = I(V_t)\frac{W_t^{1-\gamma}}{1-\gamma} = \exp\left(Q_0 + Q_1 V_t + Q_2 \log V_t\right)\frac{W_t^{1-\gamma}}{1-\gamma}. \quad (13)$$

The investor's optimal consumption-wealth ratio and the optimal dynamic option-based portfolio insurance strategy are:

$$\frac{C_t}{W_t} = \beta, \quad (14)$$

$$n_t = \frac{1}{\gamma}\frac{[(\mu - r)S_t(1 + p_s) + \lambda\sigma p_v]F_t}{[S_t^2 + S_t^2 p_s^2 + 2S_t^2 p_s + \sigma^2 p_v^2 + 2\rho(\sigma S_t p_v + p_s\sigma S_t p_v)]V_t}$$

$$+\frac{1}{\gamma}\left(Q_1 + Q_2\frac{1}{V_t}\right)\frac{\sigma\left(\rho S_t + \rho S_t p_s + \sigma p_v\right)F_t}{S_t^2 + S_t^2 p_s^2 + 2S_t^2 p_s + \sigma^2 p_v^2 + 2\rho(\sigma S_t p_v + p_s\sigma S_t p_v)}. \quad (15)$$

For the time being, we defer solving this model since this solution is merely a special case of our model setting when $\varphi = 1$. In the next section we will use perturbation methods to find the general solution to our model.

In fact, a postulated pricing kernel that has been chosen targets to obtain a parametric specification of the price dynamics for the put options. Unfortunately, the intertemporal consumption and portfolio choice problem is hard to solve in closed form, and this multi-period portfolio choice problem sometimes can be solved numerically only. However, the result of being without a closed-form solution does not come from the assumption of the pricing kernel and the assumption of the price dynamics for the put options, while is a result from the more general assumption of the investor's preference and the time-varying investment opportunities. Comparing with the numerical solutions, approximate analytical solutions - which are based on perturbations of known exact solutions - can offer economics insights into investor behavior in models that fall outside the still limited class that can be solved exactly. In other words, we can show directly which factors and how these factors affect the choosing of the optimal dynamic option-based portfolio insurance strategies from the analytical solution instead of the numerical solutions. This kind of solution can also offer more economics insights into our model. In addition, we will provide a calibration exercise in the paper to illustrate the economic results of the paper, and more economics analyses of our solution are presented in section 4 for the reader.

3.2 Approximate closed-form solution by perturbation methods

The basic idea behind the use of perturbation methods is that of formulating a general problem, on the condition that we find a particular case that has a known solution, and then using that particular case and its solution as a starting point for computing approximate solutions to nearby problems. In many financial economic models, determining the unknown function plays a key role in economic analysis under the assumption of a given functional form. However, the more generalized the model is, the more difficult it is to find a closed-form solution, especially in the case of an intertemporal consumption and portfolio choice problem with stochastic non-linear partial differential equations. In spite of this, this situation has very recently begun to change as a result of several related developments. One of these developments involves the use of perturbation methods in some special cases where solutions are derived for computing approximate solutions that will help make economic analysis more explicit. These methods offer analytical insights into investor behavior in models that fall outside the still-limited class that can be solved exactly (Campbell, 2000).

Judd and Guu (1997, 2000), Kogan and Uppal (2001), Campbell and Viceira (1999, 2001 and 2002), and Chacko and Viceira (2005) etc. use this approach to solve dynamic economic or financial models. In the remainder of this paper, we apply perturbation methods to solve our model. In the context of our problem, the insight we obtain is that the solution for the recursive utility function when $\varphi = 1$ provides a convenient starting point for performing the expansion. We apply $\varphi = 1$ in the previous section as our starting point and compute our model around this solution.

Without the restriction of $\varphi = 1$, the Bellman equation can be expressed as the following equation by substituting equation (10) into equation (12) and conjecturing there exists a solution of the functional form $J(W_t, V_t) = I(V_t)\frac{W_t^{1-\gamma}}{1-\gamma}$:

$$0 = -\frac{\beta^{\varphi}}{1-\varphi} I^{1+\frac{1-\varphi}{1-\gamma}} + \frac{\varphi}{1-\varphi}\beta\, I + Ir + I_V(\frac{1}{1-\gamma})\kappa(\theta - V_t)$$

$$+\frac{1}{2}\frac{1}{\gamma} I \frac{\left((\mu - r)S_t(1+p_s) + \lambda\sigma p_v\right)^2}{[S_t^{\ 2} + S_t^{\ 2}p_s^2 + 2S_t^{\ 2}p_s + \sigma^2 p_v^{\ 2} + 2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)]V_t}$$

$$+\frac{1}{2}\frac{1}{\gamma}\frac{(I_V)^2}{I}\frac{\sigma^2\left(\rho S_t + \rho S_t p_s + \sigma p_v\right)^2 V_t}{S_t^{\ 2} + S_t^{\ 2}p_s^2 + 2S_t^{\ 2}p_s + \sigma^2 p_v^{\ 2} + 2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)}$$

$$+\frac{1}{2} I_{VV}\frac{1}{1-\gamma}\sigma^2 V_t + \frac{1}{\gamma} I_V \frac{[(\mu - r)S_t(1+p_s) + \lambda\sigma p_v]\sigma\left(\rho S_t + \rho S_t p_s + \sigma p_v\right)}{S_t^{\ 2} + S_t^{\ 2}p_s^2 + 2S_t^{\ 2}p_s + \sigma^2 p_v^{\ 2} + 2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)}. \quad (16)$$

To simplify, we can make the transformation $I(V_t) = \Phi(V_t)^{-\frac{1-\gamma}{1-\varphi}}$ and give the following non-homogeneous ordinary differential equation:

$$0 = -\beta^{\varphi}\Phi^{-1} + \varphi\beta + (1-\varphi)r - \frac{\Phi_V}{\Phi}\kappa(\theta - V_t) - \frac{1}{2}\sigma^2 V_t\left[\left(\frac{\gamma - 1}{1-\varphi} - 1\right)\left(\frac{\Phi_V}{\Phi}\right)^2 + \frac{\Phi_{VV}}{\Phi}\right]$$

$$+(1-\varphi)\frac{1}{2}\frac{1}{\gamma}\frac{[(\mu - r)S_t(1+p_s) + \lambda\sigma p_v]^2}{[S_t^{\ 2} + S_t^{\ 2}p_s^2 + 2S_t^{\ 2}p_s + \sigma^2 p_v^{\ 2} + 2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)]V_t}$$

$$+\frac{1}{2}\frac{1}{\gamma}\frac{(\gamma-1)^2}{1-\varphi}\left(\frac{\Phi_V}{\Phi}\right)^2\frac{\sigma^2\left(\rho S_t + \rho S_t p_s + \sigma p_v\right)^2 V_t}{S_t^{\ 2} + S_t^{\ 2}p_s^2 + 2S_t^{\ 2}p_s + \sigma^2 p_v^{\ 2} + 2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)}$$

$$+\frac{\gamma - 1}{\gamma}\frac{[(\mu - r)S_t(1+p_s) + \lambda\sigma p_v]\sigma\left(\rho S_t + \rho S_t p_s + \sigma p_v\right)}{S_t^{\ 2} + S_t^{\ 2}p_s^2 + 2S_t^{\ 2}p_s + \sigma^2 p_v^{\ 2} + 2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)}\frac{\Phi_V}{\Phi}. \quad (17)$$

The above equation unfortunately cannot be computed in closed form. Our approach is to obtain an asymptotic approximation to equation (17), where the expansion is by taking a log-linear expansion of the consumption-wealth ratio around its unconditional mean as shown in the papers of Campbell (1993), Campbell and Viceira (1999, 2001 and 2002) and Chacko and Viceira (2005). From the transformation $I(V_t) = \Phi(V_t)^{-\frac{1-\gamma}{1-\varphi}}$, we get the envelope condition of the equation (10):

$$\frac{C_t}{W_t} = \beta^{\varphi}\Phi^{-1} = \exp\{\log(\frac{C_t}{W_t})\} \equiv \exp\{c_t - w_t\}. \quad (18)$$

Using a first-order Taylor expansion of $\exp\{c_t - w_t\}$ around the expectation of $(c_t - w_t)$, we can write:

$$\beta^{\varphi}\Phi^{-1} \approx \exp\{E(c_t - w_t)\} + \exp\{E(c_t - w_t)\}\cdot\left[(c_t - w_t) - E(c_t - w_t)\right]$$

$$= \exp\{E(c_t - w_t)\}\cdot\{1 - E(c_t - w_t)\} + \exp\{E(c_t - w_t)\}\cdot(c_t - w_t)$$

$$\equiv \phi_0 + \phi_1(c_t - w_t). \quad (19)$$

We now substitute equation (19) into equation (17) and guess this equation has a solution of the form $\Phi(V_t) = \exp(\hat{Q}_0 + \hat{Q}_1 V_t + \hat{Q}_2 \log V_t)$, and from this guessed solution, equation (18) can find that:

$$(c_t - w_t) = \log\{\beta^{\varphi}[\exp(\hat{Q}_0 + \hat{Q}_1 V_t + \hat{Q}_2 \log V_t)]^{-1}\}$$

$$= \varphi\log\beta - \hat{Q}_0 - \hat{Q}_1 V_t - \hat{Q}_2 \log V_t. \quad (20)$$

As such, we can express equation (17) as:

$$0 = -\left\{\phi_0 + \phi_1\left[\varphi\log\beta - \hat{Q}_0 - \hat{Q}_1 V_t - \hat{Q}_2\left(\log\theta + \frac{1}{\theta}V_t - 1\right)\right]\right\} + \varphi\beta + (1-\varphi)r - \left(\hat{Q}_1 + \hat{Q}_2\frac{1}{V_t}\right)\kappa(\theta - V_t)$$

$$+ (1-\varphi)\frac{1}{2}\frac{1}{\gamma}\frac{[(\mu - r)S_t(1+p_s) + \lambda\sigma p_v]^2}{S_t^{\,2} + S_t^{\,2}p_s^2 + 2S_t^{\,2}p_s + \sigma^2 p_v^{\,2} + 2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)}\frac{1}{V_t}$$

$$+ \frac{1}{2}\frac{1}{\gamma}\frac{(1-\gamma)^2}{1-\varphi}\left(\hat{Q}_1 + \hat{Q}_2\frac{1}{V_t}\right)^2\frac{\sigma^2\left(\rho S_t + \rho S_t p_s + \sigma p_v\right)^2}{S_t^{\,2} + S_t^{\,2}p_s^2 + 2S_t^{\,2}p_s + \sigma^2 p_v^{\,2} + 2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)}V_t$$

$$+ \frac{1}{2}\sigma^2 V_t\left[\left(\frac{1-\gamma}{1-\varphi} + 1\right)\left(\hat{Q}_1 + \hat{Q}_2\frac{1}{V_t}\right)^2 - \left(\hat{Q}_1 + \hat{Q}_2\frac{1}{V_t}\right)^2 + \hat{Q}_2\frac{1}{V_t^2}\right]$$

$$+ \frac{(\gamma - 1)}{\gamma}\frac{[(\mu - r)S_t(1+p_s) + \lambda\sigma p_v]\sigma\left(\rho S_t + \rho S_t p_s + \sigma p_v\right)}{S_t^{\,2} + S_t^{\,2}p_s^2 + 2S_t^{\,2}p_s + \sigma^2 p_v^{\,2} + 2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)}\left(\hat{Q}_1 + \hat{Q}_2\frac{1}{V_t}\right). \quad (21)$$

Rearranging the above equation, we have the following three equations for $\hat{Q}_2$, $\hat{Q}_1$, and $\hat{Q}_0$:

$$\left[\frac{1}{2\gamma}\frac{(\gamma-1)^2}{1-\varphi}\frac{\sigma^2(\rho S_t+\rho S_t p_s+\sigma p_v)^2}{S_t^{\ 2}+S_t^{\ 2}p_s^2+2S_t^{\ 2}p_s+\sigma^2 p_v^{\ 2}+2\rho(\sigma S_t p_v+p_s\sigma S_t p_v)}+\frac{1}{2}\sigma^2\frac{1-\gamma}{1-\varphi}\right]\hat{Q}_2^2$$

$$+\left[\frac{1}{2}\sigma^2-\kappa\theta+\frac{\gamma-1}{\gamma}\frac{[(\mu-r)S_t(1+p_s)+\lambda\sigma p_v]\sigma(\rho S_t+\rho S_t p_s+\sigma p_v)}{S_t^{\ 2}+S_t^{\ 2}p_s^2+2S_t^{\ 2}p_s+\sigma^2 p_v^{\ 2}+2\rho(\sigma S_t p_v+p_s\sigma S_t p_v)}\right]\hat{Q}_2$$

$$+(1-\varphi)\frac{1}{2\gamma}\frac{[(\mu-r)S_t(1+p_s)+\lambda\sigma p_v]^2}{[S_t^{\ 2}+S_t^{\ 2}p_s^2+2S_t^{\ 2}p_s+\sigma^2 p_v^{\ 2}+2\rho(\sigma S_t p_v+p_s\sigma S_t p_v)]}=0, \qquad (22)$$

$$\left[\frac{1}{2}\frac{1-\gamma}{1-\varphi}\sigma^2+\frac{1}{2}\frac{(1-\gamma)^2}{\gamma}\frac{1}{1-\varphi}\frac{\sigma^2(\rho S_t+\rho S_t p_s+\sigma p_v)^2}{S_t^{\ 2}+S_t^{\ 2}p_s^2+2S_t^{\ 2}p_s+\sigma^2 p_v^{\ 2}+2\rho(\sigma S_t p_v+p_s\sigma S_t p_v)}\right]\hat{Q}_1^2$$

$$+(\phi_1+\kappa)\hat{Q}_1+\phi_1\frac{1}{\theta}\hat{Q}_2=0, \qquad (23)$$

$$\left[\frac{1-\gamma}{1-\varphi}\sigma^2+\frac{(1-\gamma)^2}{\gamma}\frac{1}{1-\varphi}\frac{\sigma^2(\rho S_t+\rho S_t p_s+\sigma p_v)^2}{S_t^{\ 2}+S_t^{\ 2}p_s^2+2S_t^{\ 2}p_s+\sigma^2 p_v^{\ 2}+2\rho(\sigma S_t p_v+p_s\sigma S_t p_v)}\right]\hat{Q}_1\hat{Q}_2$$

$$-\left[\frac{1-\gamma}{\gamma}\frac{[(\mu-r)S_t(1+p_s)+\lambda\sigma p_v]\sigma(\rho S_t+\rho S_t p_s+\sigma p_v)}{S_t^{\ 2}+S_t^{\ 2}p_s^2+2S_t^{\ 2}p_s+\sigma^2 p_v^{\ 2}+2\rho(\sigma S_t p_v+p_s\sigma S_t p_v)}+\kappa\theta\right]\hat{Q}_1$$

$$+(\phi_1\log\theta-\phi_1+\kappa)\hat{Q}_2+\phi_1\hat{Q}_0-\phi_0-\phi_1\varphi\log\beta+\varphi\beta+(1-\varphi)r=0, \qquad (24)$$

where $\hat{Q}_2$ can be solved to the quadratic equation (22), $\hat{Q}_1$ can be solved to equation (23) given $\hat{Q}_2$, and $\hat{Q}_0$ can be solved to equation (24), given $\hat{Q}_2$ and $\hat{Q}_1$.

We can now get the indirect utility function and the optimal consumption rule and the optimal dynamic option-based portfolio insurance strategy in the stochastic environment without constraint when $\varphi=1$. The indirect utility function is:

$$J(W_t,V_t)=I(V_t)\frac{W_t^{1-\gamma}}{1-\gamma}=\Phi(V_t)^{-\frac{1-\gamma}{1-\varphi}}\frac{W_t^{1-\gamma}}{1-\gamma}$$

$$=\exp\left[-\left(\frac{1-\gamma}{1-\varphi}\right)(\hat{Q}_0+\hat{Q}_1V_t+\hat{Q}_2\log V_t)\right]\frac{W_t^{1-\gamma}}{1-\gamma}. \qquad (25)$$

The investor's optimal instantaneous consumption-wealth ratio is:

$$\frac{C_t}{W_t} = \beta^{\varphi} \exp\left(-\hat{Q}_0 - \hat{Q}_1 V_t - \hat{Q}_2 \log V_t\right). \tag{26}$$

The optimal dynamic option-based portfolio insurance strategy is:

$$n_t = \frac{1}{\gamma} \frac{[(\mu - r)S_t(1+p_s) + \lambda\sigma p_v]F_t}{[S_t^{\,2} + S_t^{\,2} p_s^2 + 2S_t^{\,2} p_s + \sigma^2 p_v^{\,2} + 2\rho(\sigma S_t p_v + p_s \sigma S_t p_v)]V_t}$$

$$+\left(1-\frac{1}{\gamma}\right)\left(\frac{\hat{Q}_1}{1-\varphi} + \frac{\hat{Q}_2}{1-\varphi}\frac{1}{V_t}\right)\frac{(\rho\sigma S_t + \rho p_s \sigma S_t + p_v \sigma^2)F_t}{S_t^{\,2} + S_t^{\,2} p_s^2 + 2S_t^{\,2} p_s + \sigma^2 p_v^{\,2} + 2\rho(\sigma S_t p_v + p_s \sigma S_t p_v)} . \tag{27}$$

Now we have explicitly solved the problem of the dynamic option-based portfolio insurance strategy for long-horizon investors with time-varying volatility. In the next section we will provide analyses of our results. In addition, we will also show the figure results of the calibration exercise in Section 4 and provide more economic explanations of the result.

4. ANALYSES OF THE OPTIMAL DYNAMIC OPTION-BASED PORTFOLIO INSURANCE STRATEGY

The purpose of this paper is in providing an investment strategy for one special kind of investor, portfolio insurers, such as some institutional investors. This kind of investment strategy is called portfolio insurance strategy and consists basically of simultaneously buying a stock (generally a financial index) and a put written on it. The more general setting of our model with time-varying volatility and recursive preference cannot be solved in closed form by solving a non-linear differential equation on the intertemporal hedging portfolio. It means that there is no exact analytical solution to it.

We can still find an approximate analytical solution following the methods described in Campbell and Viceira (2001), Kogan and Uppal (2001), and Chacko and Viceira (2005). These solutions are based on perturbations of known exact solutions. They offer analytical insights into investor behavior in models that fall outside the still limited class that can be solved exactly (Campbell, 2000). It first finds the known exact solutions - a solution for the special case in which the elasticity of substitution (φ) is equal to 1. When $\varphi = 1$, there is an exact analytical solution to our model.

We do, however, wish to address the general case of our model, where the investor's elasticity of intertemporal substitution of consumption can take any value (including, of course, when φ is equal to 1). The general case is economically meaningful for two reasons. First, it is empirically relevant, since estimates of φ available from both aggregate data and disaggregate data on individual investors suggest that φ diverges from one (Hall 1988, Campbell and Mankiw 1989, Campbell 1999, Vissin-Jorgensen 2001). Second, it nests as a special case the time-additive power utility case that is standard in the literature, because the power utility is just a special case of the recursive utility function when the elasticity of the intertemporal substitution parameter is just the inverse of the relative risk aversion coefficient. The case when the elasticity of substitution also equals one does not nest power utility unless we restrict ourselves to the special case of log utility - where both the elasticity of the intertemporal substitution parameter and the inverse of the relative risk aversion coefficient are equal to one (Chacko and Viceira (2005).

The approximate analytical solution to the general problem provides strong economic intuition about the nature of the optimal dynamic option-based portfolio insurance choice with time-varying risk, and it converges to the exact solution in those special cases when such a solution is known. Campbell (1993) and Campbell and Viceira (2002) note that the log-linear approximation solution method accurately provides that the log consumption-wealth ratio is not too variable around its unconditional mean. For the case when the elasticity of substitution equals one, this ratio is constant, and the solution is exact. Chacko and Viceira (2005) show that for all other cases, this is reasonably accurate, even for values of the elasticity of intertemporal substitution being far from one, which is in line with the findings of Campbell (1993), Campbell and Koo (1997), and Campbell et al. (2002) for the case in which risk premia and the interest rate vary over time.

The optimal dynamic option-based portfolio insurance strategy can be separated into two components: the myopic component and the intertemporal hedging component as equation (27) and in Figure 1. First, the dependence of the myopic component is simple. It is an affine function of the reciprocal of the time-varying volatility and decreases with the coefficient of relative risk aversion. Since volatility is time varying, the myopic component is time varying, too. In other words, the myopic component is simply linked to the risk-and-return tradeoff associated with the price risk of the portfolio value under the option-based portfolio insurance.

The intertemporal hedging component of the optimal dynamic option-based portfolio insurance is an affine function of the reciprocal of the time-varying volatility, with coefficient $\frac{\hat{Q}_1}{1-\varphi}$ and $\frac{\hat{Q}_2}{1-\varphi}$. While $\hat{Q}_2$ is the solution to the quadratic equation (22), $\hat{Q}_1$ is the solution to the equation (23), given $\hat{Q}_2$, $\hat{Q}_0$ is the solution to equation (24), given $\hat{Q}_1$ and $\hat{Q}_2$. When $\gamma > 1$ for the coefficient $\hat{Q}_2$, equation (22) has two real roots of opposite signs according to the quadratic equation theory. The value function J is maximized only with the solution associated with the negative root of the discriminant of the quadratic equation (22), i.e. the positive root of equation (22). It can immediately be shown that $\frac{\hat{Q}_2}{1-\varphi} > 0$.

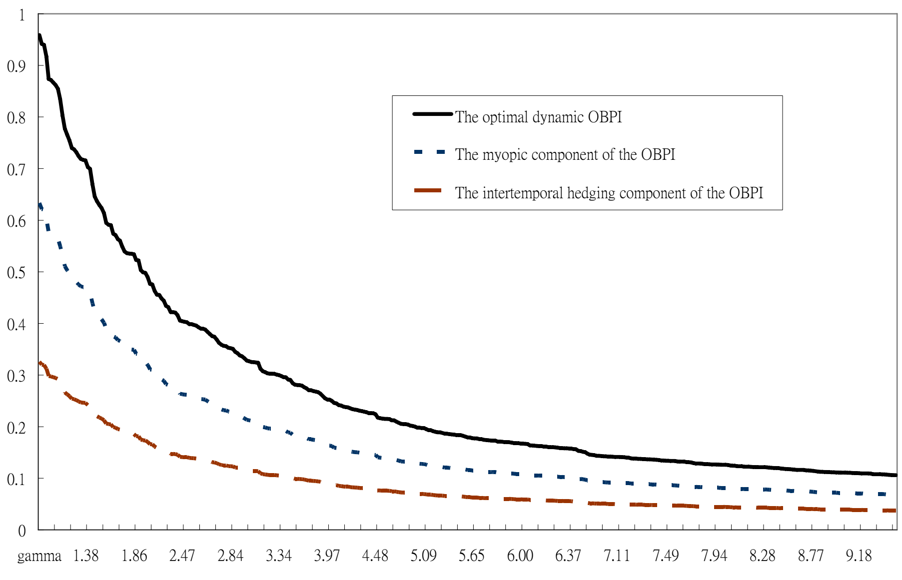

Figure 1. The optimal dynamic OBPI and its components in relation to γ.

Since $\frac{\hat{Q}_2}{1-\varphi} > 0$, it means that the sign of the coefficient of the intertemporal hedging demand coming from pure changes in time-varying volatility is positive when $\gamma > 1$. We can further separate the intertemporal hedging demand into three effects as the second component of equation (27) and in Figure 2. First, if we do not introduce any put options to create option-based portfolio insurance and instead hold only risky stock, then the intertemporal hedging component for the risky stock will consist of only the correlation effect or leverage effect ($\rho\sigma$). The intertemporal hedging component of the optimal asset allocation for risky stock is affected by the instantaneous correlation between the unexpected return and changes in stochastic volatility of the risky stock (ρ). If $\rho < 0$, then the unexpected return on the risky asset is low (the market situation is bad), and then the states of the market uncertainty will be high. Since $\frac{\hat{Q}_2}{1-\varphi} > 0$ when $\gamma > 1$, the negative instantaneous correlation between unexpected return on the risky stock and its stochastic volatility implies the investor will have negative intertemporal hedging demand due to changes solely in the volatility of the risky asset, which lacks the hedging ability against an increase in volatility. Similar discussions are found in Liu (2001) and Chacko and Viceira (2005). However, in our generalized model the consideration of introducing a put option as an option-based portfolio insurance strategy complicates the intertemporal hedging component.

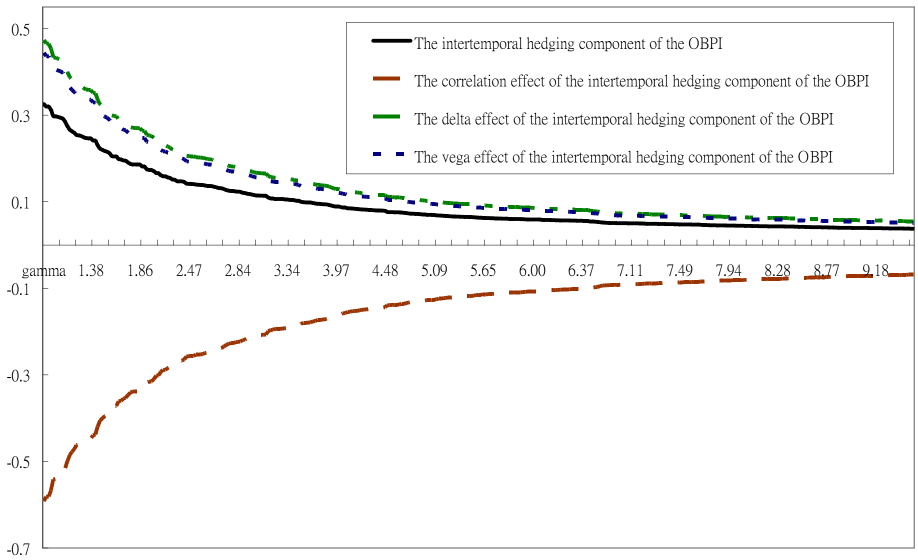

Figure 2. The intertemporal hedging demand of the optimal dynamic OBPI and its components in relation to γ.

In the previous section we assume a put option whose price exposure is negative ($p_s < 0$), and volatility exposure is positive ($p_v > 0$), without any loss of generality. From that, we show that under the leverage effect from the negative correlation between volatility of the risky stock and its price shock ($\rho < 0$), we will have two positive effects in the intertemporal hedging component for the option-based portfolio insurance: the delta effect ($\rho p_s \sigma > 0$) and the vega effect ($p_v \sigma^2 > 0$). This implies that under the correlation effect (i.e. when the unexpected return on the risky stock is low (the market situation is bad) and the market uncertainty is high), the low unexpected return on the risky stock and the high uncertainty of the market states due to the high volatility of the risky stock will make a put option play a important role in the option-based portfolio insurance due to the delta effect and the vega effect, and a conservative investor will have a positive position on the intertemporal hedging demand of the option-based portfolio insurance strategy coming from these two effects by purchasing put options.

If a conservative investor does not introduce any put options into the option-based portfolio insurance and holds only the risky stock, then she will decrease the holdings of the risky stock via the intertermporal hedging component due to the leverage effect under high volatility accompany with low unexpected return on the risky stock. In this state, when future investment opportunities are worse, put options in the intertemporal hedging component of the option-based portfolio insurance become a more valuable hedging instrument for conservative long-term investors to hedge investment-opportunity risk coming from changes in the stochastic volatility. The relationships between these components with the degree of risk averse (γ) are also seen in Figure 2.

If one does not hold any put options, then the assumption of imperfect instantaneous correlation between risky stock returns and its stochastic volatility means that the intertemporal hedging component of the risky stock can only provide partial hedging ability for multi-period investors when facing the time-varying investment opportunity set. When we introduce non-redundant put options written on the risky stock in the incomplete financial market to create option-based portfolio insurance strategies, the put options in the option-based portfolio insurance can provide differential exposure to the imperfect instantaneous correlation between volatility and stock returns, thus making the market complete. The intertemporal option-based portfolio insurance strategy can supplement the deficient hedging ability of the intertemporal hedging component of the risky asset, because of the non-linear nature of put options. Merton (1971, 1973) shows that dynamic hedging is necessary for forward-looking investors when investment opportunities are time-varying. In this paper we show that incorporating options' considerations in portfolio decisions to create a dynamic option-based portfolio insurance strategy brings benefits of improvements to the hedging ability in the intertemporal hedging component, especially during down markets.

5. CONCLUSIONS

In this paper we apply the intertemporal investment-consumption technique to discuss and explore the optimal dynamic option-based portfolio insurance strategy with stochastic volatility. We set up a model in which a long-term investor chooses an optional dynamic option-based portfolio insurance strategy by maximizing a utility function defined over intermediate consumption rather than terminal wealth, because many long-term investors desire to seek a stable consumption path with a long horizon. We show that the optimal dynamic option-based portfolio insurance strategy can be separated into two components: the myopic component and the intertemporal hedging component. The myopic component is simply linked to the risk-and-return tradeoff associated with price risk of the portfolio value under the option-based portfolio insurance. The intertemporal hedging component of the optimal dynamic option-based portfolio insurance is an affine function of the reciprocal of the time-varying volatility.

We further separate the intertemporal hedging demand into three effects. Without introducing any put options to create option-based portfolio insurance, and holding only the risky stock, the intertemporal hedging component for the risky stock will consist of only the correlation effect. The negative instantaneous correlation between unexpected return on the risky stock and its stochastic volatility implies the investor will have negative intertemporal hedging demand due to changes solely in the volatility of the risky stock, because of its lack of hedging ability against an increase in volatility. Under the leverage effect, we have two positive effects in the intertemporal hedging component for the dynamic option-based portfolio insurance: the delta effect and the vega effect. From these two effects, a conservative investor will have a positive position on the intertemporal hedging demand of the option-based portfolio insurance strategy. This means when future investment opportunities are worse, put options in the intertemporal hedging component of the option-based portfolio insurance become a more valuable hedging instrument to hedge investment-opportunity risk coming from changes in the stochastic volatility.

Merton (1971, 1973) shows that dynamic hedging is necessary for forward-looking investors when investment opportunities are time-varying. In this paper we further show that incorporating options' considerations in portfolio decisions to create a dynamic option-based portfolio insurance strategy brings benefits of improvements to the hedging ability in the intertemporal hedging component, especially during down markets.

APPENDIX A

The derivation of the special case for an optimal dynamic option-based portfolio insurance strategy when $\varphi = 1$

We conjecture there exists a solution of the functional form $J(W_t, V_t) = I(V_t)\frac{W_t^{1-\gamma}}{1-\gamma}$ when $\varphi = 1$, and substitute it into equation (12):

$$0 = \left(\log\beta - \frac{1}{1-\gamma}\log I - 1\right)\beta\, I + Ir + I_V\left(\frac{1}{1-\gamma}\right)\kappa(\theta - V_t)$$

$$+\frac{1}{2}\frac{1}{\gamma}I\frac{\left((\mu-r)S_t(1+p_s)+\lambda\sigma p_v\right)^2}{[S_t^{\,2}+S_t^{\,2}p_s^2+2S_t^{\,2}p_s+\sigma^2 p_v^{\,2}+2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)]V_t}$$

$$+\frac{1}{2}\frac{1}{\gamma}\frac{(I_V)^2}{I}\frac{\sigma^2\left(\rho S_t+\rho S_t p_s+\sigma p_v\right)^2 V_t}{S_t^{\,2}+S_t^{\,2}p_s^2+2S_t^{\,2}p_s+\sigma^2 p_v^{\,2}+2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)}$$

$$+\frac{1}{2}I_{VV}\frac{1}{1-\gamma}\sigma^2 V_t+\frac{1}{\gamma}I_V\frac{[(\mu-r)S_t(1+p_s)+\lambda\sigma p_v]\sigma\left(\rho S_t+\rho S_t p_s+\sigma p_v\right)}{S_t^{\,2}+S_t^{\,2}p_s^2+2S_t^{\,2}p_s+\sigma^2 p_v^{\,2}+2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)}\,. \quad \text{(A1)}$$

The above ordinary differential equation has a solution of the form $I = \exp(Q_0 + Q_1 V_t + Q_2\log V_t)$, so (A1) can be expressed as:

$$0 = \left\{\log\beta - \frac{1}{1-\gamma}\left[Q_0 + Q_1V_t + Q_2\left(\log\theta + \frac{1}{\theta}V_t - 1\right)\right] - 1\right\}\beta + r + \frac{1}{1-\gamma}\kappa(Q_1\theta - Q_1V_t + \frac{Q_2}{V_t}\theta - Q_2)$$

$$+\frac{1}{2}\frac{1}{\gamma}\frac{\left((\mu-r)S_t(1+p_s)+\lambda\sigma p_v\right)^2}{[(S_t)^2+(S_t)^2p_s^2+2(S_t)^2p_s+\sigma^2(p_v)^2+2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)]V_t}$$

$$+\frac{1}{2}\frac{1}{\gamma}\frac{\sigma^2\left(\rho S_t+\rho S_t p_s+\sigma p_v\right)^2}{(S_t)^2+(S_t)^2p_s^2+2(S_t)^2p_s+\sigma^2(p_v)^2+2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)}(Q_1^2V_t + 2Q_1Q_2 + \frac{Q_2^2}{V_t})$$

$$+\frac{1}{2}\sigma^2\frac{1}{1-\gamma}(Q_1^2V_t + 2Q_1Q_2 + \frac{Q_2^2}{V_t} - \frac{Q_2}{V_t})$$

$$+\frac{1}{\gamma}\left(Q_1 + \frac{Q_2}{V_t}\right)\frac{\left((\mu-r)S_t(1+p_s)+\lambda\sigma p_v\right)\sigma\left(\rho S_t+\rho S_t p_s+\sigma p_v\right)}{(S_t)^2+(S_t)^2p_s^2+2(S_t)^2p_s+\sigma^2(p_v)^2+2\rho\left(\sigma S_t p_v + p_s\sigma S_t p_v\right)}\,. \quad \text{(A2)}$$

Rearranging the above equation, we have the following three equations for Q_2, Q_1 and

Q_0:

$$\left[\frac{1}{2}\sigma^2\frac{1}{1-\gamma}+\frac{1}{2}\frac{1}{\gamma}\frac{\sigma^2\left(\rho S_t+\rho S_t p_s+\sigma p_v\right)^2}{S_t^{\,2}+S_t^{\,2}p_s^2+2S_t^{\,2}p_s+\sigma^2 p_v^{\,2}+2\rho\left(\sigma S_t p_v+p_s\sigma S_t p_v\right)}\right]Q_2^2$$

$$+\left[\frac{1}{1-\gamma}\kappa\theta-\frac{1}{2}\frac{1}{1-\gamma}\sigma^2+\frac{1}{\gamma}\frac{[(\mu-r)S_t(1+p_s)+\lambda\sigma p_v]\sigma\left(\rho S_t+\rho S_t p_s+\sigma p_v\right)}{S_t^{\,2}+S_t^{\,2}p_s^2+2S_t^{\,2}p_s+\sigma^2 p_v^{\,2}+2\rho\left(\sigma S_t p_v+p_s\sigma S_t p_v\right)}\right]Q_2$$

$$+\frac{1}{2}\frac{1}{\gamma}\frac{\left((\mu-r)S_t(1+p_s)+\lambda\sigma p_v\right)^2}{S_t^{\,2}+S_t^{\,2}p_s^2+2S_t^{\,2}p_s+\sigma^2 p_v^{\,2}+2\rho\left(\sigma S_t p_v+p_s\sigma S_t p_v\right)}=0\,, \tag{A3}$$

$$\left[\frac{1}{2}\frac{1}{1-\gamma}\sigma^2+\frac{1}{2}\frac{1}{\gamma}\frac{\sigma^2\left(\rho S_t+\rho S_t p_s+\sigma p_v\right)^2}{S_t^{\,2}+S_t^{\,2}p_s^2+2S_t^{\,2}p_s+\sigma^2 p_v^{\,2}+2\rho\left(\sigma S_t p_v+p_s\sigma S_t p_v\right)}\right]Q_1^2$$

$$-\left(\frac{1}{1-\gamma}\beta+\frac{1}{1-\gamma}\kappa\right)Q_1-\frac{1}{1-\gamma}\beta\frac{1}{\theta}Q_2=0\,, \tag{A4}$$

$$\frac{1}{1-\gamma}\left(\beta-\beta\log\theta-\kappa\right)Q_2+\left[\frac{1}{\gamma}\frac{[(\mu-r)S_t(1+p_s)+\lambda\sigma p_v]\sigma\left(\rho S_t+\rho S_t p_s+\sigma p_v\right)}{S_t^{\,2}+S_t^{\,2}p_s^2+2S_t^{\,2}p_s+\sigma^2 p_v^{\,2}+2\rho\left(\sigma S_t p_v+p_s\sigma S_t p_v\right)}\right.$$

$$\left.+\frac{1}{1-\gamma}\kappa\theta\right]Q_1+\left[\frac{1}{\gamma}\frac{\sigma^2\left(\rho S_t+\rho S_t p_s+\sigma p_v\right)^2}{S_t^{\,2}+S_t^{\,2}p_s^2+2S_t^{\,2}p_s+\sigma^2 p_v^{\,2}+2\rho\left(\sigma S_t p_v+p_s\sigma S_t p_v\right)}+\sigma^2\frac{1}{1-\gamma}\right]Q_1Q_2$$

$$-\frac{1}{1-\gamma}\beta Q_0+\beta\log\beta-\beta+r=0\,. \tag{A5}$$

From equation (A3), we have:

$$Q_2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\,, \tag{A6}$$

where

$$a=\frac{1}{2}\sigma^2\frac{1}{1-\gamma}+\frac{1}{2}\frac{1}{\gamma}\frac{\sigma^2\left(\rho S_t+\rho S_t p_s+\sigma p_v\right)^2}{S_t^{\,2}+S_t^{\,2}p_s^2+2S_t^{\,2}p_s+\sigma^2 p_v^{\,2}+2\rho\left(\sigma S_t p_v+p_s\sigma S_t p_v\right)}$$

$$b=\frac{1}{1-\gamma}\kappa\theta-\frac{1}{2}\frac{1}{1-\gamma}\sigma^2+\frac{1}{\gamma}\frac{[(\mu-r)S_t(1+p_s)+\lambda\sigma p_v]\sigma\left(\rho S_t+\rho S_t p_s+\sigma p_v\right)}{S_t^{\,2}+S_t^{\,2}p_s^2+2S_t^{\,2}p_s+\sigma^2 p_v^{\,2}+2\rho\left(\sigma S_t p_v+p_s\sigma S_t p_v\right)}$$

$$c = \frac{1}{2}\frac{1}{\gamma}\frac{\left((\mu - r)S_t(1 + p_s) + \lambda\sigma p_v\right)^2}{S_t^{\ 2} + S_t^{\ 2} p_s^2 + 2S_t^{\ 2} p_s + \sigma^2 p_v^{\ 2} + 2\rho\left(\sigma S_t p_v + p_s \sigma S_t p_v\right)}$$

From this result, we can get the indirect utility function and the optimal consumption rule and optimal dynamic option-based portfolio insurance strategy when $\varphi = 1$.

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