Optimal Dynamic Asset Allocation with Capital Gains Taxes and Stochastic Volatility

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ABSTRACT

This paper applies the option in the tax law to investigate the effect of taxation of capital gains on the optimal dynamic consumption and portfolio choice when there is predictable variation in return volatility. For a conservative investor, under the leverage effect with capital gains tax, we provide a negative after-tax leverage effect on the intertemporal hedging demand coming from pure changes of stochastic volatility with the assumption of the negative value of the instantaneous correlation between the unexpected return on the stock and its stochastic volatility. Moreover, in a bad market accompanied by high volatility under the leverage effect, a conservative investor will have a negative vega effect of the tax option on the intertemporal hedging demand coming from pure changes of stochastic volatility.

Keywords: Capital gains tax; Portfolio choice; Stochastic volatility; Intertemporal model, Intertemporal hedging demand

JEL classification: H24, G11, C61

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1. Introduction

In the recent years, there has been some research exploring the optimal dynamic asset allocation strategies with various risks, such as volatility risk, interest risk or inflation risk. Merton (1971) was the first to consider the effect of a stochastic investment opportunity set in the analysis of optimal asset allocation strategies for long-horizon investors. However, a vast empirical literature in the 1990's has demonstrated that some degree of asset return is predictable. Bollerslev, Chou and Kroner (1992), Campbell, Lo and MacKinlay (1997), and Campbell, Lettau, Malkiel and Xu (2001) have shown that stock market return volatility is not constant over time. Since then, academic economists have emerged studying the effects of return predictability on asset allocation strategies. Brennan, Schwartz and Lagnado (1997) and some of the recent research for this area explores models which examine the optimal dynamic asset allocation strategy when the state variable follows stochastic processes. However, there is a very limited literature on the capital gains taxes which apply to the optimal dynamic asset allocation strategies with time-varying volatility risk.

While there are substantial differences across countries in both the level and structure of capital income taxes, investors in many countries are generally subject to a non-trivial amount of taxes. Whenever they sell securities they hold at a profit, one may think that the taxes on capital gains have an appreciable impact on an individual's consumption and investment decisions. Therefore, taxes play an important role in the decision-making process of individuals concerning their consumption and investment plans. The taxation of returns on financial assets alters the benefits of saving for future consumption and thus affects the trade-off between current consumption and investment (Dammon, Spatt and Zhang, 2001).

The purpose of this paper is applying the real option in the tax law to investigate the effect of taxation of capital gains on the optimal dynamic consumption and portfolio choice with stochastic volatility. Computing the optimal consumption and portfolio policy of an investor subject to capital gains taxes is a challenging task. Our research contributes to the literature on optimal asset allocation by exploring precisely how capital gains taxes affect asset allocation with stochastic volatility.

If asset returns or volatility are time-varying, this implies that investment opportunities are time-varying, too. Merton (1971, 1973) shows that when investment opportunities are time-varying, dynamic hedging is necessary for forward-looking investors. Multi-period or long-horizon investors are concerned not only with expected returns and risk today, but with ways in which expected returns and risk may change over time. Dynamic asset allocation strategies for multi-period or long-horizon investors differ from those of single-period investors because the former demand risky assets not only for their risk premia, but also for their hedging ability against adverse changes in future investment opportunities. Merton (1969, 1971, 1973) shows that if investment opportunities are varying overtime, then long-horizon investors generally care about shocks to investment opportunities and not just about wealth itself. They may seek to hedge their exposures to wealth shocks, and this creates intertemporal hedging demand for financial assets (Campbell, 2000).

Recently, there has been some limited literature exploring and analyzing optimal dynamic portfolio choice with volatility risk (Liu, 2000, 2001 and Chacko and Viceira, 2005). They solve for the optimal consumption and portfolio choice of long-horizon investors when there is predictable variation in stock market return volatility. However, there is no other research exploring both the effects of capital gains taxes and stochastic volatility on optimal portfolio choice. In addition, while various countries have their own tax laws, the tax laws in many countries usually create a situation where the taxpayer's payoff from a course of action resembles the payoff from writing a call option to the government. As a result of the call-like nature of the investor's tax pay-off function, investors have an incentive to reduce their expected tax burdens. This incentive will result in the adjustment of optimal dynamic asset allocation strategies and the consumption rule. In addition, other things equal, the capital gains tax system generally imposes a high burden on more volatile investments than on less volatile investments with the same expected return. In other words, the tax system also imposes a higher burden when the market is more volatile. Investors can reduce their expected tax burdens by reducing the volatility of their capital gains. One way to reduce capital gains volatility is also through intertemporal hedging on the financial assets, especially when facing an environment with time-varying volatility. We find that multi-period investors value assets not only for their short-term risk-return characteristics, but also for their ability to hedge consumption against adverse shifts in future "after-tax" investment opportunities. Thus these investors have an extra demand for risky assets that reflects after-tax intertemporal hedging.

In this generalized intertemporal model under the stochastic environment, Merton's approach (1971, 1973) could not be used to derive a closed-form solution by solving a nonlinear differential equation on the intertemporal hedging portfolio. Recently however, some of the literature has begun to work on it, such as the approximate analytical solutions developed by Campbell and Viceira (2001), Kogan and Uppal (2001), and Chacko and Viceira (2005). These solutions are based on perturbations of known exact solutions. They offer analytical insights into investor behavior in models that fall outside the still limited class that can be solved exactly (Campbell, 2000). In this paper, we use perturbation methods to get linear approximate solutions. We mainly derived the explicit solution on a log-linear expansion of the consumption-wealth ratio around its unconditional mean provided by Campbell (1993), Campbell and Viceira (1999, 2001 and 2002) and Chacko and Viceira (2005).

This paper is organized as follows. Section II describes the model used and environment assumed in this paper. Section III develops the model of optimal consumption policy and dynamic asset allocation strategies with time-varying volatility and capital gains taxes. Section IV provides analyses of the model results and how capital gains taxes affect asset allocation with stochastic volatility. Finally, conclusions are given in Section V.

2. The Model

2.1 Investment Opportunity Set

In this paper, we assume that the investor invests wealth in tradable assets only. There are two tradable assets available for trading in the economy. One of the assets is a riskless money market fund, denoted by B_t with a constant interest rate of r. Its instantaneous return is

$$\frac{dB_t}{B_t} = rdt \ . \tag{1}$$

The short rate is assumed to be constant and tax-free in order to focus on the stochastic volatility of the risky asset. The second tradable asset is a taxable risky stock. S_t denotes the price of the risky financial asset at time t; its instantaneous total return dynamics are given by

$$\frac{dS_t}{S_t} = \mu\, dt + \sqrt{V_t}\, dZ_S \,, \tag{2}$$

where μ is the instantaneous expected rate of return on the risky stock; and $\sqrt{V_t}$ is the time-varying instantaneous standard deviation of the return on the risky asset. We denote stochastic variables with a subscript “t”; and let the conditional variance of the risky stock vary stochastically over time. From the following setting, the investment opportunity is time-varying. We assume that the instantaneous variance process is

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dZ_v , \qquad (3)$$

where the parameter $\theta > 0$, which describes the long-term mean of the variance, $\kappa \in (0,1)$ is the reversion parameter of the instantaneous variance process, i.e. this parameter describes the degree of mean reversion. dZ_S and dZ_v are two Wiener processes with constant correlation ρ. We assume that the stock returns are correlated with changes in volatility with instantaneous correlation ρ, which may be assumed to be negative to capture the leverage effect or the asymmetric effect (Glosten et al., 1993). The negative correlation assumption with the mean-reversion on stock returns volatility can capture two of the most important features discussed in the empirical literature on the equity market.

Each monetary unit of stock sold at some time t is subject to the payment of an amount of tax computed according to the relative tax basis observed at the prior time. This paper assumes that the tax laws create a situation where the tax payer's payoff from a course of action resembles the payoff from holding a call option. As a result of the call-like nature of the taxpayer's tax function, and following Sircar and Papanicolaou (1999) and Liu and Pan (2003) and under the above setting, we assume the value of the real tax option ($T_t = \tau(S_t, V_t)$) which is the function (τ) on the prices of the stock (S_t) and on the volatility of stock returns (V_t) at time t, and will have the following price dynamics:

$$dT_t = [(\mu - r)S_t\tau_s + \lambda\sigma\tau_v V_t + rT_t]\, dt + \sqrt{\mathrm{V}_t}S_t\tau_s dZ_s + \sigma\sqrt{V_t}\tau_v dZ_v , \qquad (4)$$

where λ determines the stochastic volatility risk premium of the real tax option, and $0 < \tau_s < 1$ and $\tau_v > 0$ are measures of the real tax option's price sensitivity to small changes in the underlying stock price and volatility, respectively. They measure the sensitivity of the real tax call option value to infinitesimal changes in the stock price and volatility, respectively. Specifically,

$$\tau_s = \left.\frac{\partial \tau(s,v)}{\partial s}\right|_{(S_t,V_t)} ; \quad \tau_v = \left.\frac{\partial \tau(s,v)}{\partial v}\right|_{(S_t,V_t)} .$$

The real tax option written on the stock with the non-linear payoff structure $\tau(S_\delta, V_\delta) = (S_\delta - K)^+$ for some strike price $K > 0$, at $t < \delta$, and the strike price K, in fact, is the investor's tax basis for the risky asset observed at the prior time. Thus, the dynamics of the after-tax return on the risky stock (S_t^e) would be

$$dS_t^e = \left[\mu S_t - (\mu - r) S_t \tau_s - \lambda \sigma \, \tau_v V_t - r T_t\right] dt + \left[\sqrt{V_t} S_t - \sqrt{V_t} S_t \tau_s\right] dZ_s - \sigma \sqrt{V_t} \tau_v dZ_v . \qquad (5)$$

2.2 Preferences

We assume that the investor's preference is recursive and of the form described by Duffie and Epstein (1992). Recursive utility is a generalization of the standard and time-separable power utility function that separates the elasticity of intertemporal substitution of consumption from the relative risk aversion (Duffie and Epstein, 1992, Chacko and Viceira, 2005). This means that the power utility is just a special case of the recursive utility function when the elasticity of the intertemporal substitution is just the inverse of the relative risk aversion coefficient.

$$J = E_t\left[\int_t^\infty f(C_\delta, J_\delta)\, d\delta\right], \qquad (6)$$

where $f(C_\delta, J_\delta)$ is a normalized aggregator of investor's current consumption (C_δ) and utility has the following form:

$$f(C,J) = \beta\left(1 - \frac{1}{\varphi}\right)^{-1}(1-\gamma) J \left[\left(\frac{C}{((1-\gamma)J)^{\frac{1}{1-\gamma}}} \right)^{1-\frac{1}{\varphi}} - 1 \right], \qquad (7)$$

where γ is the coefficient of relative risk aversion, β is the rate of time preference and φ is the elasticity of intertemporal substitution; they are all larger than zero.

The investor's objective is to maximize her expected lifetime utility by choosing consumption and the proportions of her wealth to invest in the two tradable assets subject to the following intertemporal budget constraint,

$$dW_t = \left[n_t \left(\mu \frac{S_t}{S_t^e} - (\mu - r)\frac{S_t}{S_t^e}\tau_s - \lambda\sigma\frac{\tau_v}{S_t^e}V_t - \frac{rT_t}{S_t^e} - r \right) W_t + rW_t - C_t \right] dt$$

$$+ n_t \left(\sqrt{V_t}\frac{S_t}{S_t^e} - \sqrt{V_t}\frac{S_t}{S_t^e}\tau_s \right) dZ_s W_t - n_t \left(\sigma\sqrt{V_t}\frac{\tau_v}{S_t^e} \right) dZ_v W_t , \quad (8)$$

where W_t represents the investor's total wealth, while n_t are the fractions of the investor's financial wealth allocated to the risky stock at time t, and C_t represents the investor's instantaneous consumption.

3. Optimal Consumption Policy and Dynamic Asset Allocation Strategies with Time-Varying Volatility and Capital Gains Taxes

The main objection of this paper is to explore the optimal dynamic asset allocation strategies with time-varying volatility and capital gains taxes. Instead of a single period result, we also want to explore the optimal intertemporal consumption with after-tax stochastic investment opportunity set induced by the stochastic volatility.

3.1 A Special Case with Unit Elasticity of Intertemporal Substitution of Consumption

The value function of the problem (J) is to maximize the investor's expected lifetime utility. The principle of optimality leads to the following Bellman equation for the utility function. Under the above setting, the Bellman equation will satisfy

$$0=\sup_{n,C}\left\{f(C_\delta,J_\delta)+J_W\left[n_t\left(\mu\frac{S_t}{S_t^e}-(\mu-r)\frac{S_t}{S_t^e}\tau_s-\lambda\sigma\frac{\tau_v}{S_t^e}V_t-\frac{rT_t}{S_t^e}-r\right)W_t+rW_t-C_t\right]\right.$$

$$+J_V\left[\kappa(\theta-V_t)\right]+\frac{1}{2}J_{WW}\left[n_t^2\left(\frac{S_t}{S_t^e}-\frac{S_t}{S_t^e}\tau_s\right)^2V_t+n_t^2\left(\sigma\frac{\tau_v}{S_t^e}\right)^2V_t-2n_t^2V_t\left(\frac{S_t}{S_t^e}-\frac{S_t}{S_t^e}\tau_s\right)\left(\sigma\frac{\tau_v}{S_t^e}\right)\rho\right]W_t^2$$

$$\left.+\frac{1}{2}J_{VV}\sigma^2V_t+J_{WV}W_t\left[n_t\left(\frac{S_t}{S_t^e}-\frac{S_t}{S_t^e}\tau_s\right)V_t\sigma\rho-n_t\left(\frac{\tau_v}{S_t^e}\right)\sigma^2V_t\right]\right\}, \tag{9}$$

where J_W, J_V. denote the derivatives of J with respect to wealth, W, and stochastic volatility, V_t, respectively. We will use the similar notation for higher derivatives as well. We also note that ρ is the instantaneous correlation between the unexpected return on the stock and its stochastic volatility.

The first-order conditions for the equation (9) are

$$C_t=J_W^{-\varphi}J^{\frac{1-\varphi\gamma}{1-\gamma}}\beta^{\varphi}(1-\gamma)^{\frac{1-\varphi\gamma}{1-\gamma}}, \tag{10}$$

$$n_t=-\frac{J_W}{J_{WW}W_t}\frac{[(\mu-r)S_t(1-\tau_s)-\lambda\sigma\tau_vV_t]\,S_t^e}{[S_t^{\,2}+S_t^{\,2}\tau_s^2-2S_t^{\,2}\tau_s+\sigma^2\tau_v^{\,2}-2\rho(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v)]\,V_t}$$

$$-\frac{J_{WV}}{J_{WW}W_t}\frac{\sigma\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)S_t^e}{S_t^{\,2}+S_t^{\,2}\tau_s^2-2S_t^{\,2}\tau_s+\sigma^2\tau_v^{\,2}-2\rho(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v)}. \tag{11}$$

The optimal dynamic asset allocation strategy has two major components. The first term is the mean-variance portfolio weight. This is for an investor who only invests in a single period horizon or under constant investment opportunity set, the myopic demand. The second term of the optimal dynamic portfolio allocation is the intertemporal hedging demand that characterizes demand arising from the desire to hedge against changes in the after-tax investment opportunity set induced by the stochastic volatility. This term is determined by the instantaneous rate of changes in relation to the value function.

We will discuss this in more detail later, because the first-order conditions for our problem are not explicit solutions unless we know the complicated indirect utility function. Substituting the first-order solutions back into the Bellman equation, we get

$$0 = f(C(J),J) - J_W C(J) + J_W r W_t + J_V\left[\kappa(\theta - V_t)\right] + \frac{1}{2} J_{VV}\sigma^2 V_t$$

$$-\frac{1}{2}\frac{(J_W)^2}{J_{WW}}\frac{\left((\mu-r)S_t(1-\tau_s)-\lambda\sigma\tau_v V_t\right)^2}{[S_t{}^2+S_t{}^2\tau_s^2-2S_t{}^2\tau_s+\sigma^2\tau_v{}^2-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)]V_t}$$

$$-\frac{1}{2}\frac{(J_{WV})^2}{J_{WW}}\frac{\sigma^2\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)^2 V_t}{S_t{}^2+S_t{}^2\tau_s^2-2S_t{}^2\tau_s+\sigma^2(\tau_v)^2-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}$$

$$-\frac{J_{WV}J_W}{J_{WW}}\frac{\left((\mu-r)S_t(1-\tau_s)-\lambda\sigma\tau_v V_t\right)\sigma\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)}{S_t{}^2+S_t{}^2\tau_s^2-2S_t{}^2\tau_s+\sigma^2\tau_v{}^2-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}. \qquad (12)$$

We conjecture that there exists a solution of the functional form $J(W_t,V_t) = I(V_t)\frac{W_t^{1-\gamma}}{1-\gamma}$ when $\varphi = 1$, and substitute it into equation (12), then the ordinary differential equation will have a solution of the form $I = \exp(Q_0 + Q_1 V_t + Q_2 \log V_t)$. Rearranging that equation, we have three equations for Q_2, Q_1 and Q_0 after collecting terms in $\frac{1}{V_t}$, V_t and 1. We provide the full details in Appendix.

We are now able to obtain the indirect utility function and the optimal consumption rule and dynamic asset allocation strategy with time-varying volatility and capital gains tax when $\varphi = 1$. The indirect utility function is

$$J(W_t,V_t) = I(V_t)\frac{W_t^{1-\gamma}}{1-\gamma} = \exp\left(Q_0 + Q_1 V_t + Q_2 \log V_t\right)\frac{W_t^{1-\gamma}}{1-\gamma}. \qquad (13)$$

The investor's optimal consumption-wealth ratio and the optimal dynamic asset allocation strategy are

$$\frac{C_t}{W_t} = \beta, \qquad (14)$$

$$n_t = \frac{1}{\gamma}\frac{[(\mu - r)S_t(1-\tau_s) - \lambda\sigma\tau_v V_t]S_t^e}{[S_t^{\,2} + S_t^{\,2}\tau_s^2 + 2S_t^{\,2}\tau_s + \sigma^2\tau_v^{\,2} - 2\rho(\sigma S_t\tau_v - \tau_s\sigma S_t\tau_v)]V_t}$$

$$+\frac{1}{\gamma}\left(Q_1 + Q_2\frac{1}{V_t}\right)\frac{\sigma\left(\rho S_t - \rho S_t\tau_s - \sigma\tau_v\right)S_t^e}{S_t^{\,2} + S_t^{\,2}\tau_s^2 - 2S_t^{\,2}\tau_s + \sigma^2\tau_v^{\,2} - 2\rho(\sigma S_t\tau_v - \tau_s\sigma S_t\tau_v)}. \qquad (15)$$

However, for the time being, we defer solving this model since this solution is merely a special case of our model setting when $\varphi = 1$. In the next section, we will use perturbation methods to find the general solution to our model.

3.2 Approximate Closed-Form Solution by Perturbation Methods

The basic idea behind the use of perturbation methods is that of formulating a general problem, on the condition that we find a particular case that has a known solution, and then using that particular case and its solution as a starting point for computing approximate solutions to nearby problems. In many financial economic models, determining the unknown function plays a key role in economic analysis under the assumption of a given functional form. However, the more generalized the model is, the more difficult it is to find a closed-form solution, especially in the case of an intertemporal consumption and portfolio choice problem with stochastic nonlinear partial differential equations. In spite of this, this situation has very recently begun to change as a result of several related developments. One of these developments has involved the use of perturbation methods in some special cases where solutions are derived for computing approximate solutions that will help make economic analysis more explicit. These methods offer analytical insights into investor behavior in models that fall outside the still-limited class that can be solved exactly (Campbell, 2000).

Judd and Guu (1997, 2000), Kogan and Uppal (2001), Campbell and Viceira (1999, 2001 and 2002), and Chacko and Viceira (2005) etc. have used this approach to solve dynamic economic or financial models. In the remainder of this paper, we will apply perturbation methods to solve our model. In the context of our problem, the insight we obtain is that the solution for the recursive utility function when $\varphi = 1$ provides a convenient starting point for performing the expansion. We apply the $\varphi = 1$ in the previous section as our starting point and compute our model around this solution.

Without the restriction of $\varphi = 1$, the Bellman equation can be expressed as the following equation by substituting equation (10) into equation (12) and conjecturing there exists a solution of the functional form $J(W_t, V_t) = I(V_t)\frac{W_t^{1-\gamma}}{1-\gamma}$,

$$0 = -\frac{\beta^{\varphi}}{1-\varphi} I^{1+\frac{1-\varphi}{1-\gamma}} + \frac{\varphi}{1-\varphi}\beta I + Ir + I_V(\frac{1}{1-\gamma})\kappa(\theta - V_t)$$

$$+\frac{1}{2}\frac{1}{\gamma} I \frac{\left((\mu - r)S_t(1-\tau_s) - \lambda\sigma\tau_v V_t\right)^2}{[S_t^{\,2} + S_t^{\,2}\tau_s^2 - 2S_t^{\,2}\tau_s + \sigma^2\tau_v^{\,2} - 2\rho\left(\sigma S_t\tau_v - \tau_s\sigma S_t\tau_v\right)]V_t}$$

$$+\frac{1}{2}\frac{1}{\gamma}\frac{\left(I_V\right)^2}{I}\frac{\sigma^2\left(\rho S_t - \rho S_t\tau_s - \sigma\tau_v\right)^2 V_t}{S_t^{\,2} + S_t^{\,2}\tau_s^2 - 2S_t^{\,2}\tau_s + \sigma^2\tau_v^{\,2} - 2\rho\left(\sigma S_t\tau_v - \tau_s\sigma S_t\tau_v\right)}$$

$$+\frac{1}{2} I_{VV}\frac{1}{1-\gamma}\sigma^2 V_t + \frac{1}{\gamma} I_V \frac{[(\mu - r)S_t(1-\tau_s) - \lambda\sigma\tau_v V_t]\sigma\left(\rho S_t - \rho S_t\tau_s - \sigma\tau_v\right)}{S_t^{\,2} + S_t^{\,2}\tau_s^2 - 2S_t^{\,2}\tau_s + \sigma^2\tau_v^{\,2} - 2\rho\left(\sigma S_t\tau_v - \tau_s\sigma S_t\tau_v\right)}. \tag{16}$$

To simplify, we can make the transformation $I(V_t) = \Phi(V_t)^{-\frac{1-\gamma}{1-\varphi}}$, and give the following non-homogeneous ordinary differential equation,

$$0 = -\beta^{\varphi}\Phi^{-1} + \varphi\beta + (1-\varphi)r - \frac{\Phi_V}{\Phi}\kappa(\theta - V_t) - \frac{1}{2}\sigma^2 V_t\left[\left(\frac{\gamma-1}{1-\varphi} - 1\right)\left(\frac{\Phi_V}{\Phi}\right)^2 + \frac{\Phi_{VV}}{\Phi}\right]$$

$$+(1-\varphi)\frac{1}{2}\frac{1}{\gamma}\frac{[(\mu - r)S_t(1-\tau_s) - \lambda\sigma\tau_v V_t]^2}{[S_t^{\,2} + S_t^{\,2}\tau_s^2 - 2S_t^{\,2}\tau_s + \sigma^2\tau_v^{\,2} - 2\rho\left(\sigma S_t\tau_v - \tau_s\sigma S_t\tau_v\right)]V_t}$$

$$+\frac{1}{2}\frac{1}{\gamma}\frac{(\gamma-1)^2}{1-\varphi}\left(\frac{\Phi_V}{\Phi}\right)^2\frac{\sigma^2\left(\rho S_t - \rho S_t\tau_s - \sigma\tau_v\right)^2 V_t}{S_t^{\,2} + S_t^{\,2}\tau_s^2 - 2S_t^{\,2}\tau_s + \sigma^2\tau_v^{\,2} - 2\rho\left(\sigma S_t\tau_v - \tau_s\sigma S_t\tau_v\right)}$$

$$+\frac{\gamma-1}{\gamma}\frac{[(\mu - r)S_t(1-\tau_s) - \lambda\sigma\tau_v V_t]\sigma\left(\rho S_t - \rho S_t\tau_s - \sigma\tau_v\right)}{S_t^{\,2} + S_t^{\,2}\tau_s^2 - 2S_t^{\,2}\tau_s + \sigma^2\tau_v^{\,2} - 2\rho\left(\sigma S_t\tau_v - \tau_s\sigma S_t\tau_v\right)}\frac{\Phi_V}{\Phi}. \tag{17}$$

Unfortunately, the above equation cannot be computed in closed form. Our approach is to obtain an asymptotic approximation to equation (17), where the expansion is by taking a log-linear expansion of the consumption-wealth ratio around its unconditional mean as shown in the papers of Campbell (1993), Campbell and Viceira (1999, 2001 and 2002) and Chacko and Viceira (2005). From the transformation $I(V_t) = \Phi(V_t)^{-\frac{1-\gamma}{1-\varphi}}$, we can get the envelope condition of the equation (10),

$$\frac{C_t}{W_t} = \beta^{\varphi}\Phi^{-1} = \exp\{\log(\frac{C_t}{W_t})\} \equiv \exp\{c_t - w_t\} . \tag{18}$$

Then, using a first-order Taylor expansion of $\exp\{c_t - w_t\}$ around the expectation of $(c_t - w_t)$, we can write

$$\begin{aligned}\beta^{\varphi}\Phi^{-1} &\approx \exp\{E(c_t - w_t)\} + \exp\{E(c_t - w_t)\} \cdot \left[(c_t - w_t) - E(c_t - w_t)\right] \\ &= \exp\{E(c_t - w_t)\} \cdot \{1 - E(c_t - w_t)\} + \exp\{E(c_t - w_t)\} \cdot (c_t - w_t) \\ &\equiv \phi_0 + \phi_1 (c_t - w_t) . \end{aligned} \tag{19}$$

Substituting equation (19) into the equation (17) and guessing this equation has a solution of the form $\Phi(V_t) = \exp(\hat{Q}_0 + \hat{Q}_1 V_t + \hat{Q}_2 \log V_t)$, and from this guessed solution, equation (18) can find that

$$\begin{aligned}(c_t - w_t) &= \log\{\beta^{\varphi}[\exp(\hat{Q}_0 + \hat{Q}_1 V_t + \hat{Q}_2 \log V_t)]^{-1}\} \\ &= \varphi \log\beta - \hat{Q}_0 - \hat{Q}_1 V_t - \hat{Q}_2 \log V_t . \end{aligned} \tag{20}$$

As such, we can express equation (17) as

$$0=-\left\{\phi_0+\phi_1\left[\varphi\log\beta-\hat{Q}_0-\hat{Q}_1V_t-\hat{Q}_2\left(\log\theta+\frac{1}{\theta}V_t-1\right)\right]\right\}+\varphi\beta+(1-\varphi)r$$

$$-\left(\hat{Q}_1+\hat{Q}_2\frac{1}{V_t}\right)\kappa(\theta-V_t)+(1-\varphi)\frac{1}{2}\frac{1}{\gamma}\frac{[(\mu-r)S_t(1-\tau_s)-\lambda\sigma\tau_vV_t]^2}{S_t^{\,2}+S_t^{\,2}\tau_s^2-2S_t^{\,2}\tau_s+\sigma^2\tau_v^{\,2}-2\rho(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v)}\frac{1}{V_t}$$

$$+\frac{1}{2}\frac{1}{\gamma}\frac{(1-\gamma)^2}{1-\varphi}\left(\hat{Q}_1+\hat{Q}_2\frac{1}{V_t}\right)^2\frac{\sigma^2(\rho S_t-\rho S_t\tau_s-\sigma\tau_v)^2}{S_t^{\,2}+S_t^{\,2}\tau_s^2-2S_t^{\,2}\tau_s+\sigma^2\tau_v^{\,2}-2\rho(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v)}V_t$$

$$+\frac{1}{2}\sigma^2V_t\left[\left(\frac{1-\gamma}{1-\varphi}+1\right)\left(\hat{Q}_1+\hat{Q}_2\frac{1}{V_t}\right)^2-\left(\hat{Q}_1+\hat{Q}_2\frac{1}{V_t}\right)^2+\hat{Q}_2\frac{1}{V_t^2}\right]$$

$$+\frac{(\gamma-1)}{\gamma}\frac{[(\mu-r)S_t(1-\tau_s)-\lambda\sigma\tau_vV_t]\sigma(\rho S_t-\rho S_t\tau_s-\sigma\tau_v)}{S_t^{\,2}+S_t^{\,2}\tau_s^2-2S_t^{\,2}\tau_s+\sigma^2\tau_v^{\,2}-2\rho(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v)}\left(\hat{Q}_1+\hat{Q}_2\frac{1}{V_t}\right). \qquad (21)$$

Rearranging the above equation, we have the following three equations for $\hat{Q}_2$, $\hat{Q}_1$ and $\hat{Q}_0$,

$$\left[\frac{1}{2\gamma}\frac{(\gamma-1)^2}{1-\varphi}\frac{\sigma^2(\rho S_t-\rho S_t\tau_s-\sigma\tau_v)^2}{S_t^{\,2}+S_t^{\,2}\tau_s^2-2S_t^{\,2}\tau_s+\sigma^2\tau_v^{\,2}-2\rho(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v)}+\frac{1}{2}\sigma^2\frac{1-\gamma}{1-\varphi}\right]\hat{Q}_2^2$$

$$+\left[\frac{1}{2}\sigma^2-\kappa\theta+\frac{\gamma-1}{\gamma}\frac{[(\mu-r)S_t(1-\tau_s)]\sigma(\rho S_t-\rho S_t\tau_s-\sigma\tau_v)}{S_t^{\,2}+S_t^{\,2}\tau_s^2-2S_t^{\,2}\tau_s+\sigma^2\tau_v^{\,2}-2\rho(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v)}\right]\hat{Q}_2$$

$$+(1-\varphi)\frac{1}{2\gamma}\frac{[(\mu-r)S_t(1-\tau_s)]^2}{[S_t^{\,2}+S_t^{\,2}\tau_s^2-2S_t^{\,2}\tau_s+\sigma^2\tau_v^{\,2}-2\rho(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v)]}=0, \qquad (22)$$

$$\left[\frac{1}{2}\frac{1-\gamma}{1-\varphi}\sigma^2+\frac{1}{2}\frac{(1-\gamma)^2}{\gamma}\frac{1}{1-\varphi}\frac{\sigma^2(\rho S_t-\rho S_t\tau_s-\sigma\tau_v)^2}{S_t^{\,2}+S_t^{\,2}\tau_s^2-2S_t^{\,2}\tau_s+\sigma^2\tau_v^{\,2}-2\rho(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v)}\right]\hat{Q}_1^2$$

$$+\left(\frac{\gamma-1}{\gamma}\frac{-\lambda\sigma\tau_v\sigma(\rho S_t-\rho S_t\tau_s-\sigma\tau_v)}{S_t^{\,2}+S_t^{\,2}\tau_s^2-2S_t^{\,2}\tau_s+\sigma^2\tau_v^{\,2}-2\rho(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v)}+\phi_1+\kappa\right)\hat{Q}_1+\phi_1\frac{1}{\theta}\hat{Q}_2$$

$$+(1-\varphi)\frac{1}{2\gamma}\frac{(\lambda\sigma\tau_v)^2}{S_t^{\,2}+S_t^{\,2}\tau_s^2-2S_t^{\,2}\tau_s+\sigma^2\tau_v^{\,2}-2\rho(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v)}=0, \qquad (23)$$

$$\left[\frac{1-\gamma}{1-\varphi}\sigma^2+\frac{(1-\gamma)^2}{\gamma}\frac{1}{1-\varphi}\frac{\sigma^2\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)^2}{S_t{}^2+S_t{}^2\tau_s^2-2S_t{}^2\tau_s+\sigma^2\tau_v{}^2-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}\right]\hat{Q}_1\hat{Q}_2$$

$$-\left[\frac{1-\gamma}{\gamma}\frac{[(\mu-r)S_t(1-\tau_s)]\sigma\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)}{S_t{}^2+S_t{}^2\tau_s^2-2S_t{}^2\tau_s+\sigma^2\tau_v{}^2-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}+\kappa\theta\right]\hat{Q}_1$$

$$+\left(\frac{\gamma-1}{\gamma}\frac{-\lambda\sigma\tau_v\sigma\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)}{S_t{}^2+S_t{}^2\tau_s^2-2S_t{}^2\tau_s+\sigma^2\tau_v{}^2-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}+\phi_1\log\theta-\phi_1+\kappa\right)\hat{Q}_2$$

$$+\phi_1\hat{Q}_0+(1-\varphi)\frac{1}{\gamma}\frac{-(\mu-r)(1-\tau_s)\lambda\sigma\tau_v S_t}{S_t{}^2+S_t{}^2\tau_s^2-2S_t{}^2\tau_s+\sigma^2\tau_v{}^2-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}$$

$$-\phi_0-\phi_1\varphi\log\beta+\varphi\beta+(1-\varphi)r=0, \tag{24}$$

where $\hat{Q}_2$ can be solved to the quadratic equation (22), $\hat{Q}_1$ can be solved to the equation (23) given $\hat{Q}_2$, and $\hat{Q}_0$ can be solved to the equation (24), given $\hat{Q}_2$ and $\hat{Q}_1$.

As such, we can now get the indirect utility function and the optimal consumption rule and the optimal dynamic asset allocation strategy with capital gains tax in the stochastic environment without constraint when $\varphi=1$.

The indirect utility function is

$$J(W_t,V_t)=I(V_t)\frac{W_t^{1-\gamma}}{1-\gamma}=\Phi(V_t)^{-\frac{1-\gamma}{1-\varphi}}\frac{W_t^{1-\gamma}}{1-\gamma}$$

$$=\exp\left[-\left(\frac{1-\gamma}{1-\varphi}\right)(\hat{Q}_0+\hat{Q}_1V_t+\hat{Q}_2\log V_t)\right]\frac{W_t^{1-\gamma}}{1-\gamma}. \tag{25}$$

The investor's optimal instantaneous consumption-wealth ratio is

$$\frac{C_t}{W_t}=\beta^{\varphi}\exp\left(-\hat{Q}_0-\hat{Q}_1V_t-\hat{Q}_2\log V_t\right). \tag{26}$$

The optimal dynamic asset allocation strategy with capital gains tax is

$$n_t = \frac{1}{\gamma} \frac{[(\mu - r)S_t(1-\tau_s) - \lambda\sigma\tau_v V_t]S_t^e}{[S_t^{\ 2} + S_t^{\ 2}\tau_s^2 - 2S_t^{\ 2}\tau_s + \sigma^2\tau_v^{\ 2} - 2\rho(\sigma S_t\tau_v - \tau_s\sigma S_t\tau_v)]V_t}$$

$$+\left(1-\frac{1}{\gamma}\right)\left(\frac{\hat{Q}_1}{1-\varphi} + \frac{\hat{Q}_2}{1-\varphi}\frac{1}{V_t}\right)\frac{(\rho\sigma S_t - \rho\tau_s\sigma S_t - \tau_v\sigma^2)S_t^e}{S_t^{\ 2} + S_t^{\ 2}\tau_s^2 - 2S_t^{\ 2}\tau_s + \sigma^2\tau_v^{\ 2} - 2\rho(\sigma S_t\tau_v - \tau_s\sigma S_t\tau_v)}. \qquad (27)$$

Now we have explicitly solved the problem of the dynamic asset allocation strategy for long-horizon investors with time-varying volatility and capital gains tax. In the next section, we will provide analyses of our results.

4. Analyses of the Model Results and How Capital Gains Taxes Affect Asset Allocation with Stochastic Volatility

The optimal dynamic asset allocation strategy can be separated into two components: the myopic component, and the intertemporal hedging component. First, the dependence of the myopic component is simple. It is an affine function of the reciprocal of the time-varying volatility and decreases with the coefficient of relative risk aversion. Since volatility is time varying, the myopic component is time varying, too. In other words, the myopic component is simply linked to the after-tax risk-and-return tradeoff associated with price risk. The higher the capital gains tax rate would lead to the higher delta of the real tax option (τ_s), and this will decrease the after-tax return. And hence decrease the myopic component in the optimal dynamic asset allocation for the risky stock. In addition, we know that the capital gains tax system imposes a higher burden on more volatile risky stock than on less risky stock with the same expected return. This paper shows this phenomenon by the vega of the real tax option. The higher the vega of the real tax option, (i.e. the higher the sensitivity of the tax burden to infinitesimal changes on the stock return volatility) accompanied by the $\tau_v > 0$, the higher the increase of the tax burden with respect to the increase in the stock return volatility, and the lower the after-tax return on the risky stock will be. And hence the investor will decrease the myopic component of the asset allocation on the risky stock.

The intertemporal hedging component of the optimal dynamic asset allocation is an affine function of the reciprocal of the time-varying volatility, with coefficient $\frac{\hat{Q}_1}{1-\varphi}$ and $\frac{\hat{Q}_2}{1-\varphi}$. While $\hat{Q}_2$ is the solution to the quadratic equation (22), $\hat{Q}_1$ is the solution to the equation (23) given $\hat{Q}_2$, and $\hat{Q}_0$ is the solution to the equation (24), given $\hat{Q}_1$ and $\hat{Q}_2$. When $\gamma > 1$ for the coefficient $\hat{Q}_2$, the equation (22) has two real roots of opposite signs according to the quadratic equation theory. And the value function J is maximized only with the solution associated with the negative root of the discriminant of the quadratic equation (22), i.e. the positive root of equation (22). It can immediately be shown that $\frac{\hat{Q}_2}{1-\varphi} > 0$.

Since $\frac{\hat{Q}_2}{1-\varphi} > 0$, it means that the sign of the coefficient of the intertemporal hedging demand coming from pure changes in time-varying volatility is positive when $\gamma > 1$. We can further separate the intertemporal hedging demand into three effects. First, if we don't introduce any capital gains tax consideration, and instead the holding stock is tax-free, the intertemporal hedging component for the risky stock will consist of only the correlation effect or leverage effect ($\rho\sigma$). The intertemporal hedging component of the optimal asset allocation for risky stock without capital gains tax is affected by the instantaneous correlation between the unexpected return and changes in stochastic volatility of the risky stock (ρ). If $\rho < 0$, it means that the unexpected return on the risky asset is low (the market situation is bad), and then the states of the market uncertainty will be high. Since $\frac{\hat{Q}_2}{1-\varphi} > 0$ when $\gamma > 1$, the negative instantaneous correlation between unexpected return on the risky stock and its stochastic volatility implies the investor will have negative intertemporal hedging demand due to changes solely in the volatility of the risky asset, which lacks the hedging ability against an increase in volatility. Similar discussions are found in Liu (2001) and Chacko and Viceira (2005). However, in our generalized model, the consideration of capital gains tax with time-varying volatility complicates the intertemporal hedging component on asset allocation for long-horizon investors.

In the previous section we assume a real tax option whose price exposure is positive ($\tau_s > 0$), and volatility exposure is positive ($\tau_v > 0$), without any loss of generality. From that, we show that under the leverage effect from the negative correlation between volatility of the risky stock and its price shock ($\rho < 0$), we will have two capital gains tax effects in the intertemporal hedging component for the risky stock, the tax-option delta effect ($-\rho\tau_s\sigma > 0$), and the tax-option vega effect ($-\tau_v\sigma^2 < 0$). This implies that under the correlation effect (i.e. when the unexpected return on the risky stock is low (the market situation is bad), and the market uncertainty is high), the low unexpected return on the risky stock and the high uncertainty of the market states due to the high volatility of the risky stock will make capital gains tax play a important role in the intertemporal hedging demand due to the delta effect and the vega effect, and a conservative investor will have a positive component on the intertemporal hedging demand coming from the tax-option delta effect, and a negative component on the intertemporal hedging demand coming from the tax-option vega effect. For a conservative investor, if she doesn’t impose any tax and holds only the risky stock, she will decrease the holdings of the risky stock via the intertermporal hedging component due to the leverage effect under high volatility accompanied by low unexpected return on the risky stock.

However, under the leverage effect with capital gains tax, the negative intertemporal hedging component will be partially offset by the positive delta effect of the real tax option for $0 < \tau_s < 1$. The net leverage effect, which we term "the after-tax leverage effect", on the intertemporal hedging demand coming from pure changes of stochastic volatility is $(1-\tau_s)\rho\sigma$. This component of the intertemporal hedging demand is also negative for the assumption of the negative value of the instantaneous correlation between the unexpected return on the stock and its stochastic volatility (ρ). The consideration of the capital gains tax will decrease the absolute value of this component. The positive delta effect on tax option ($-\tau_s\rho\sigma$) is intuitive because under the leverage effect, i.e. the low unexpected returns on the risky stock with the high return volatility on the risky stock, the increase of the holding of the stock will not increase tax burden, under the bad market. Therefore, the capital gains tax effect will provide some offset effect on the leverage effect of the negative intertemporal hedging demand.

However, due to the bad market accompanied by high volatility under the leverage effect, a conservative investor will have an negative vega effect of the tax option on the intertemporal hedging demand coming from pure changes of stochastic volatility ($-\tau_v\sigma^2$) for $\tau_v > 0$. This result is that when the capital gains tax imposes a high burden on more volatile investments than on less volatile investment with the same expected return, it will tend to cause investors to allocate their capital to flow away from risky stocks and toward riskless bonds. Therefore, we will have an extra negative intertemporal hedging demand from the vega effect of tax option.

5. Conclusions

Although various countries have their own tax laws, the tax laws in many countries usually create a situation where the taxpayer's payoff from a course of action resembles the payoff from writing a call option to the government. As a result of the call-like nature of the investor's tax pay-off function, investors have an incentive to reduce their expected tax burdens. This incentive will result in the adjustment of optimal dynamic asset allocation strategies and consumption rule. The purpose of this paper is applying the real option in the tax law to investigate the effect of taxation of capital gains on the optimal dynamic consumption and portfolio choice with stochastic volatility. Our research contributes to the literature on optimal asset allocation by exploring precisely how capital gains taxes affect asset allocation with stochastic volatility.

The optimal dynamic asset allocation strategy can be separated into two components: the myopic component, and the intertemporal hedging component. The myopic component is simply linked to the after-tax risk-and-return tradeoff associated with price risk. We can further separate the intertemporal hedging demand explicitly. For a conservative investor, if she doesn't impose any tax and holds only the risky stock, she will decrease the holdings of the risky stock via the intertermporal hedging component due to the leverage effect under high volatility accompanied by low unexpected return on the risky stock. However, under the leverage effect with capital gains tax, the negative intertemporal hedging component will be partially offset by the positive delta effect of the real tax option. The net leverage effect, which we call "the after-tax leverage effect" on the intertemporal hedging demand coming from pure changes of stochastic volatility is also negative under the assumption of the negative value of the instantaneous correlation between the unexpected return on the stock and its stochastic volatility.

In this paper, we show that a bad market accompanied by high volatility under the leverage effect, a conservative investor will have a negative vega effect of the tax option on the intertemporal hedging demand coming from pure changes of stochastic volatility. This result is that when the capital gains tax imposes a high burden on more volatile investments than on less volatile investments with the same expected return, this will tend to cause investors to reallocate their capital away from risky stocks and toward riskless bonds. Therefore, we will have an extra negative intertemporal hedging demand from the vega effect of tax option.

Appendix

The derivation of the special case for optimal dynamic asset allocation strategy with capital gains tax and time-varying volatility when $\varphi = 1$

We conjecture there exists a solution of the functional form $J(W_t, V_t) = I(V_t)\frac{W_t^{1-\gamma}}{1-\gamma}$ when $\varphi = 1$, and substitute it into equation (12),

$$0 = \left(\log\beta - \frac{1}{1-\gamma}\log I - 1\right)\beta I + Ir + I_V(\frac{1}{1-\gamma})\kappa(\theta - V_t)$$

$$+\frac{1}{2}\frac{1}{\gamma}I\frac{\left((\mu-r)S_t(1-\tau_s)-\lambda\sigma\tau_v V_t\right)^2}{[S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)]V_t}$$

$$+\frac{1}{2}\frac{1}{\gamma}\frac{\left(I_V\right)^2}{I}\frac{\sigma^2\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)^2 V_t}{S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}$$

$$+\frac{1}{2}I_{VV}\frac{1}{1-\gamma}\sigma^2 V_t+\frac{1}{\gamma}I_V\frac{[(\mu-r)S_t(1-\tau_s)-\lambda\sigma\tau_v V_t]\sigma\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)}{S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}. \quad \text{(A1)}$$

The above ordinary differential equation has a solution of the form $I = \exp(Q_0 + Q_1 V_t + Q_2 \log V_t)$, so (A1) can be expressed as

$$0 = \left\{\log\beta - \frac{1}{1-\gamma}\left[Q_0 + Q_1V_t + Q_2\left(\log\theta + \frac{1}{\theta}V_t - 1\right)\right] - 1\right\}\beta + r + \frac{1}{1-\gamma}\kappa(Q_1\theta - Q_1V_t + \frac{Q_2}{V_t}\theta - Q_2)$$

$$+\frac{1}{2}\frac{1}{\gamma}\frac{\left((\mu-r)S_t(1-\tau_s)-\lambda\sigma\tau_v V_t\right)^2}{[S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)]V_t}$$

$$+\frac{1}{2}\frac{1}{\gamma}\frac{\sigma^2\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)^2}{S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}(Q_1^2V_t+2Q_1Q_2+\frac{Q_2^2}{V_t})$$

$$+\frac{1}{\gamma}\left(Q_1+\frac{Q_2}{V_t}\right)\frac{\left((\mu-r)S_t(1-\tau_s)-\lambda\sigma\tau_v V_t\right)\sigma\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)}{S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}$$

$$+\frac{1}{2}\sigma^2\frac{1}{1-\gamma}(Q_1^2V_t+2Q_1Q_2+\frac{Q_2^2}{V_t}-\frac{Q_2}{V_t}). \quad \text{(A2)}$$

Rearranging the above equation, we have the following three equations for Q_2, Q_1

and Q_0,

$$\left[\frac{1}{2}\sigma^2\frac{1}{1-\gamma}+\frac{1}{2}\frac{1}{\gamma}\frac{\sigma^2\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)^2}{S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}\right]Q_2^2$$

$$+\left[\frac{1}{1-\gamma}\kappa\theta-\frac{1}{2}\frac{1}{1-\gamma}\sigma^2+\frac{1}{\gamma}\frac{[(\mu-r)S_t(1-\tau_s)]\sigma\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)}{S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}\right]Q_2$$

$$+\frac{1}{2}\frac{1}{\gamma}\frac{\left((\mu-r)S_t(1-\tau_s)\right)^2}{S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}=0, \qquad \text{(A3)}$$

$$\left[\frac{1}{2}\frac{1}{1-\gamma}\sigma^2+\frac{1}{2}\frac{1}{\gamma}\frac{\sigma^2\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)^2}{S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}\right]Q_1^2$$

$$-\left(\frac{1}{1-\gamma}\beta+\frac{1}{1-\gamma}\kappa-\frac{1}{\gamma}\frac{-\lambda\sigma\tau_v\sigma\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)}{S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}\right)Q_1 \qquad ,$$

$$-\frac{1}{1-\gamma}\beta\frac{1}{\theta}Q_2+\frac{\left(\lambda\sigma\tau_v\right)^2}{S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}=0. \qquad \text{(A4)}$$

$$\left[\frac{1}{1-\gamma}\left(\beta-\beta\log\theta-\kappa\right)+\frac{1}{\gamma}\frac{-\lambda\sigma\tau_v\sigma\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)}{S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}\right]Q_2$$

$$+\left[\frac{1}{\gamma}\frac{[(\mu-r)S_t(1-\tau_s)]\sigma\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)}{S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}+\frac{1}{1-\gamma}\kappa\theta\right]Q_1$$

$$+\left[\frac{1}{\gamma}\frac{\sigma^2\left(\rho S_t-\rho S_t\tau_s-\sigma\tau_v\right)^2}{S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}+\sigma^2\frac{1}{1-\gamma}\right]Q_1Q_2$$

$$-\frac{1}{1-\gamma}\beta Q_0+\frac{1}{\gamma}\frac{-(\mu-r)(1-\tau_s)\lambda\sigma\tau_v S_t}{S_t^{\ 2}+S_t^{\ 2}\tau_s^2-2S_t^{\ 2}\tau_s+\sigma^2\tau_v^{\ 2}-2\rho\left(\sigma S_t\tau_v-\tau_s\sigma S_t\tau_v\right)}$$

$$+\beta\log\beta-\beta+r=0. \qquad \text{(A5)}$$

From equation (A3), we have:

$$Q_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{A6}$$

where

$$a = \frac{1}{2}\sigma^2 \frac{1}{1-\gamma} + \frac{1}{2}\frac{1}{\gamma}\frac{\sigma^2\left(\rho S_t - \rho S_t \tau_s - \sigma\tau_v\right)^2}{S_t^{\ 2} + S_t^{\ 2}\tau_s^2 - 2S_t^{\ 2}\tau_s + \sigma^2\tau_v^{\ 2} - 2\rho\left(\sigma S_t \tau_v - \tau_s \sigma S_t \tau_v\right)}$$

$$b = \frac{1}{1-\gamma}\kappa\theta - \frac{1}{2}\frac{1}{1-\gamma}\sigma^2 + \frac{1}{\gamma}\frac{[(\mu - r)S_t(1-\tau_s)]\sigma\left(\rho S_t - \rho S_t \tau_s - \sigma\tau_v\right)}{S_t^{\ 2} + S_t^{\ 2}\tau_s^2 - 2S_t^{\ 2}\tau_s + \sigma^2\tau_v^{\ 2} - 2\rho\left(\sigma S_t \tau_v - \tau_s \sigma S_t \tau_v\right)}$$

$$c = \frac{1}{2}\frac{1}{\gamma}\frac{\left((\mu - r)S_t(1-\tau_s)\right)^2}{S_t^{\ 2} + S_t^{\ 2}\tau_s^2 - 2S_t^{\ 2}\tau_s + \sigma^2\tau_v^{\ 2} - 2\rho\left(\sigma S_t \tau_v - \tau_s \sigma S_t \tau_v\right)}$$

From this result, we can get the indirect utility function and the optimal consumption rule and optimal dynamic asset allocation strategy when $\varphi = 1$.

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