

## Chapter 6 Scaling Laws in Miniaturization

### 6.1 Introduction to Scaling

- **Scaling theory** is a value guide to what may work and what may not work when we start to design the world of micro.

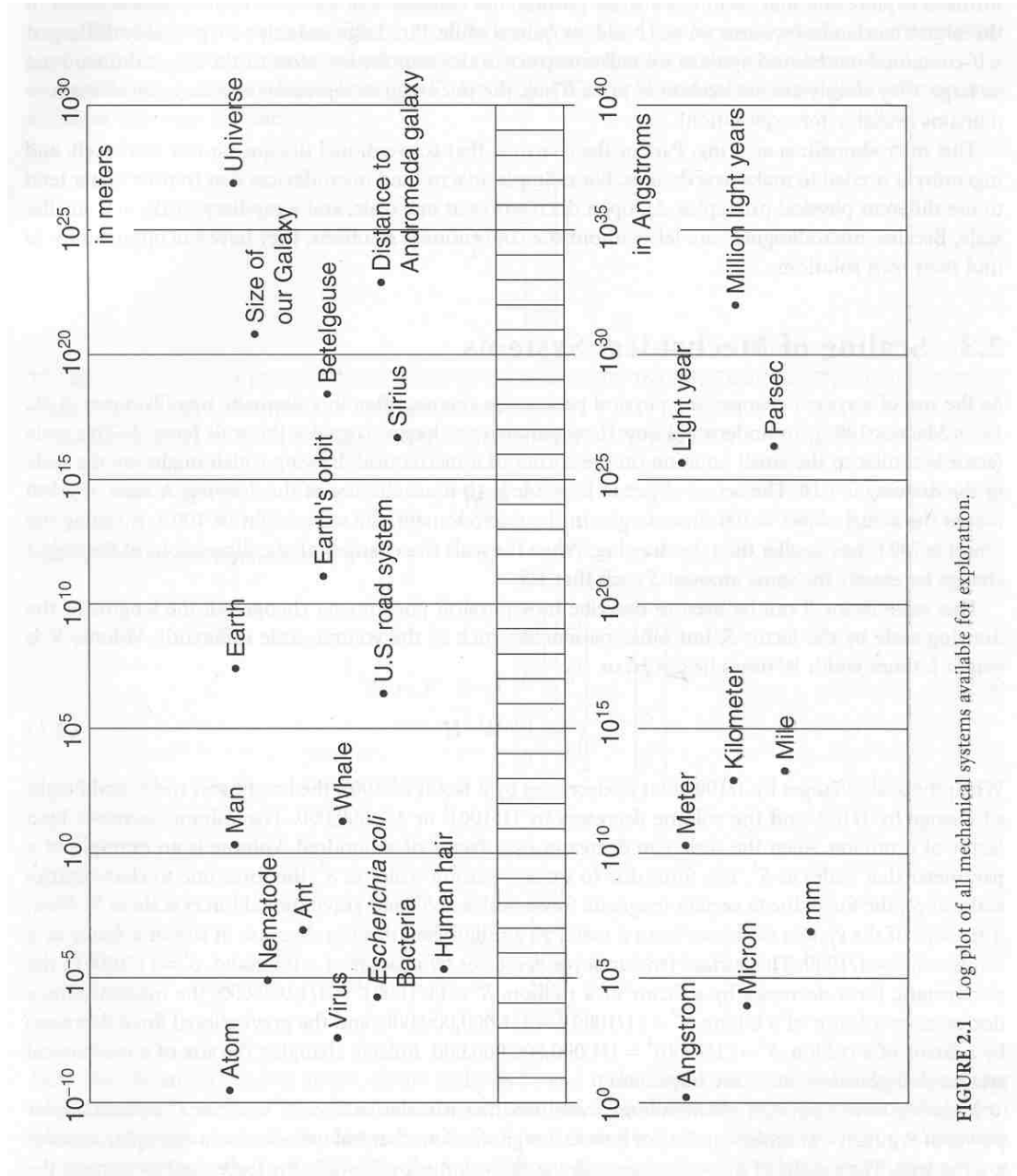


FIGURE 2.1 Log plot of all mechanical systems available for exploration.

- Three general scale sizes: (a) Astronomical (天體的) objects; (b) Macro-objects; (c) micro-objects.
  - Things effective at one of these scale sizes often are insignificant at another scale size.
  - Examples:
    - Gravitational forces dominate on an astronomical scale (e.g., the earth

moves around the sun), but not on smaller scales.

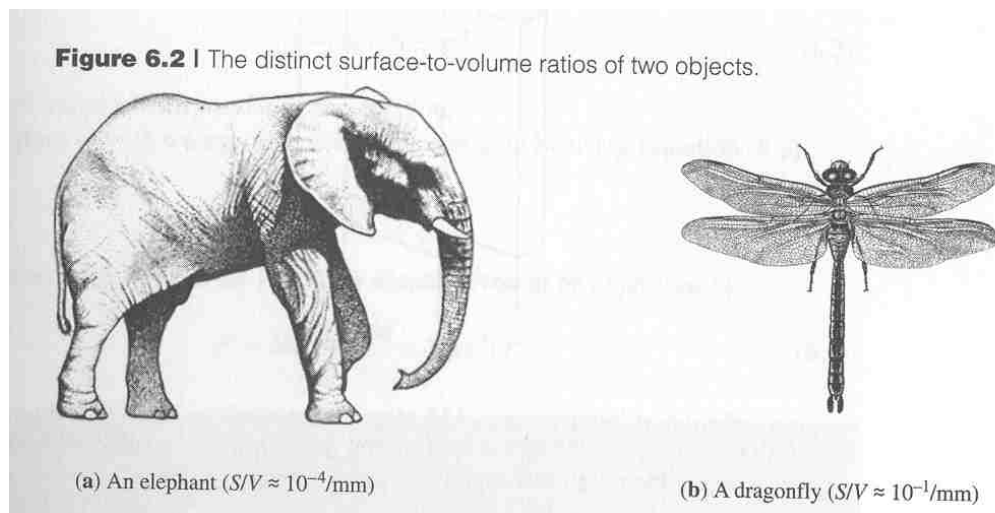
- **Macro-sized motors use magnetic forces** for actuation, but **micro-sized ones usually use electrostatic fields** instead of magnetic.

(Reference: MEMS Handbook, edited by Mohamed Gad-el-Hak, CRC Press)

- Two types of scaling laws:
  1. The **first type**: depends on the **size** of physical objects.
  2. The **second type**: involves both the **size** and **material properties** of the system.

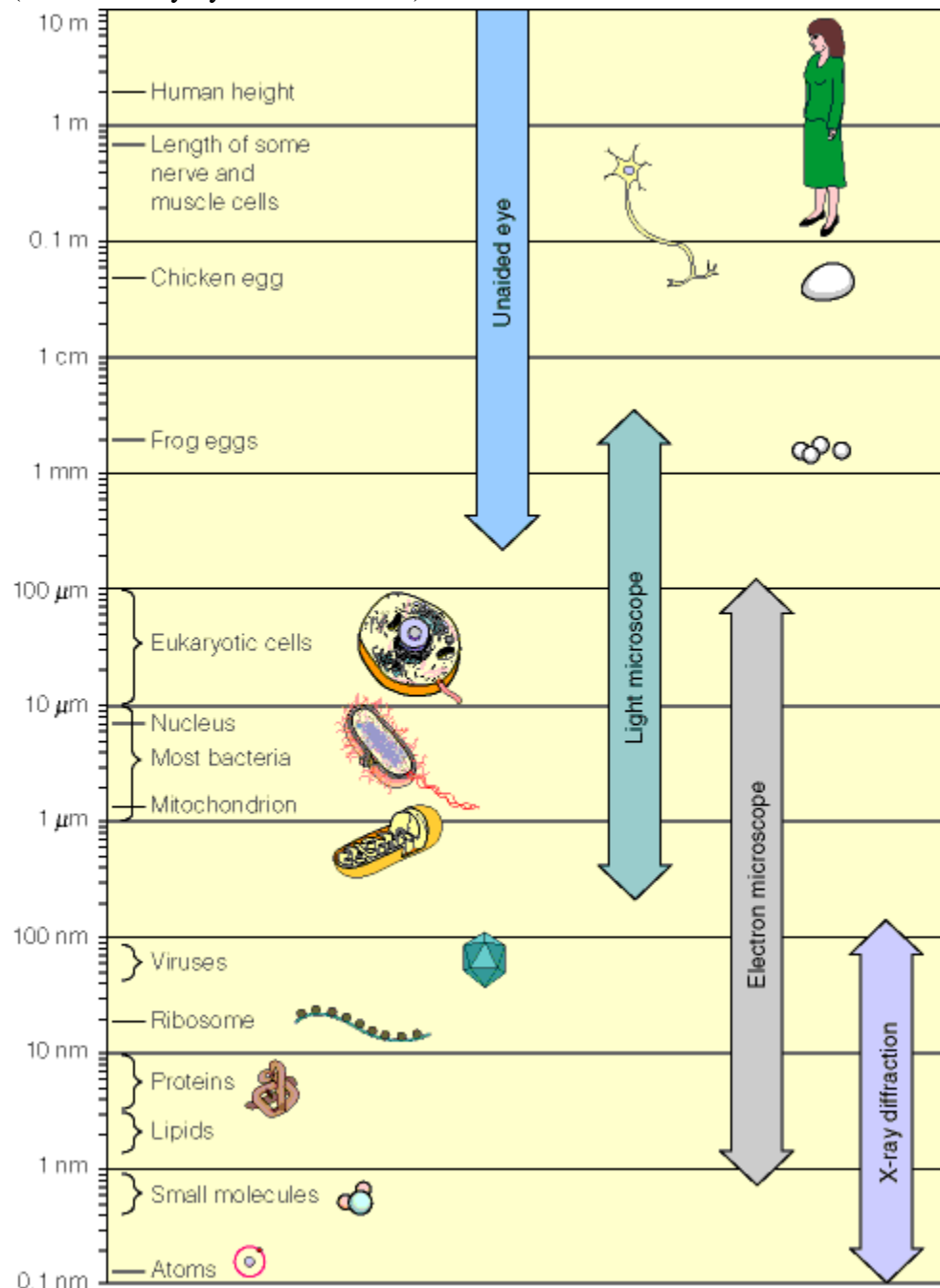
## 6.2 Scaling in Geometry

- **Surface** and **volume** are two physical quantities that are frequently involved in micro-device design.
  - **Volume**: related to the **mass** and **weight** of a device, which are related to both **mechanical and thermal inertial**.  
(**thermal inertial**: related to the heat capacity of a solid, which is a measure of how fast we can heat or cool a solid. → important in designing a thermal actuator)
  - **Surface**: related to **pressure** and the **buoyant forces** in fluid mechanics, as well as **heat absorption or dissipation** by a solid in convective heat transfer.
- Surface to volume ration ( $S/V$  ratio)
  - $S \propto l^2$ ;  $V \propto l^3$
  - $S/V \propto l^{-1}$
  - **As the size  $l$  decreases, its  $S/V$  ratio increases.**
  - Examples
    - $S/V$  ratio of an elephant ( $10^{-4}$ ) vs. of a dragonfly ( $10^{-1}$ )



- An elephant and a flea have **cells of about the same size**. Too large a cell will not have enough surface for substance exchanges with its surroundings to support the active metabolism (新陳代謝) within, unless it is highly elongated like a vertebrate nerve cell, increasing the  $S/V$  ratio. (Biochemistry by Mathews *et al.*)

Figure 1.8: Range of sizes of objects studied by biochemists and biologists. (Biochemistry by Mathews *et al.*)



**Eukaryotes** - Organisms whose cells are compartmentalized by internal cellular membranes to produce

a nucleus and organelles.

**Lipids** - 脂質,脂

### 6.3 Scaling in Rigid-Body Dynamics

#### 6.3.1 Scaling in Dynamic Forces

$$s = v_0 t + \frac{1}{2} a t^2 \quad (6.3)$$

By letting  $v_0 = 0$ ,

$$a = \frac{2s}{t^2} \quad (6.4)$$

$$F = Ma = \frac{2sM}{t^2} \propto (l)(l^3)t^{-2} \quad (6.5)$$

#### 6.3.2 The Trimmer Force Scaling Vector

- Trimmer (1989) proposed a unique matrix to represent force scaling with relative parameters of **acceleration  $a$** , **time  $t$** , and **power density  $P/V_0$** .

- **Force scaling factor:**

$$F = [l^F] = \begin{bmatrix} l^1 \\ l^2 \\ l^3 \\ l^4 \end{bmatrix} \quad (6.6)$$

- **Acceleration  $a$** : from Eq. (6.5),

$$a = [l^F][l^3]^{-1} = \begin{bmatrix} l^1 \\ l^2 \\ l^3 \\ l^4 \end{bmatrix} [l^{-3}] = \begin{bmatrix} l^{-2} \\ l^{-1} \\ l^0 \\ l^1 \end{bmatrix} \quad (6.7)$$

- **Time  $t$** : from Eq. (6.5),

$$t = \sqrt{\frac{2sM}{F}} \propto ([l^1][l^3][l^{-F}])^{0.5} = \begin{bmatrix} l^{1.5} \\ l^1 \\ l^{0.5} \\ l^0 \end{bmatrix} \quad (6.8)$$

- Power Density  $P/V_0$ : from Eq. (6.5),

Since  $P = \frac{W}{t} = \frac{F \cdot s}{t}$ , thus

$$\frac{P}{V_0} = \frac{Fs}{tV_0} \quad (6.9)$$

$$\Rightarrow \frac{P}{V_0} = \frac{[l^F][l^1]}{([l^1][l^3][l^{-F}]^{0.5}[l^3])} = \begin{bmatrix} l^{-2.5} \\ l^{-1} \\ l^{0.5} \\ l^2 \end{bmatrix} \quad (6.10)$$

**Table 6.1** | Scaling laws for rigid-body dynamics

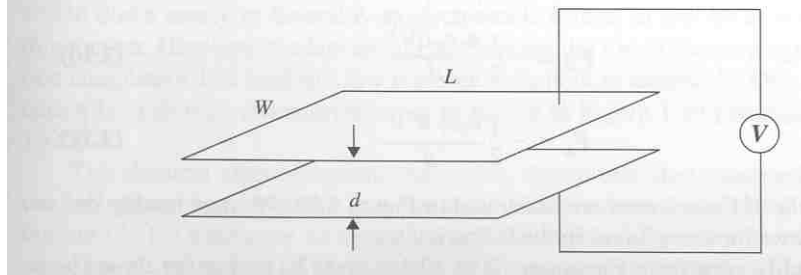
Order	Force scale $F$	Acceleration $a$	Time $t$	Power density $P/V_0$
1	$l^1$	$l^{-2}$	$l^{1.5}$	$l^{-2.5}$
2	$l^2$	$l^{-1}$	$l^1$	$l^{-1}$
3	$l^3$	$l^0$	$l^{0.5}$	$l^{0.5}$
4	$l^4$	$l^1$	$l^0$	$l^2$

## 6.4 Scaling in Electrostatic Forces

- In Fig. 6.4, the electric potential energy induced in the parallel plates is:

$$U = -\frac{1}{2}CV^2 = -\frac{\epsilon_r \epsilon_0 WL}{2d}V^2 \quad (2.7)$$

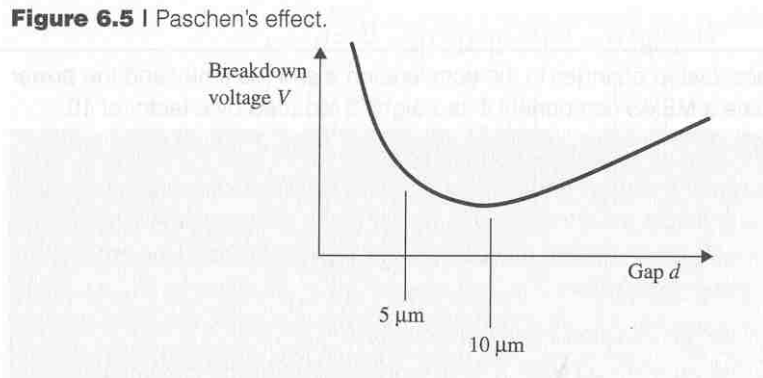
**Figure 6.4** | Electrically charged parallel plates.



- Breakdown voltage
  - The voltage required to **initiate discharge**.
  - For  $d > 10\mu\text{m}$ ,  $V \propto l^1$  (see Fig. 6.5)

$$\Rightarrow U \propto \frac{(l^0)(l^0)(l^1)(l^1)(l^1)^2}{l^1} = l^3 \quad (6.11)$$

- A factor of 10 decrease in linear dimension will decrease the potential energy by a factor of 1000.



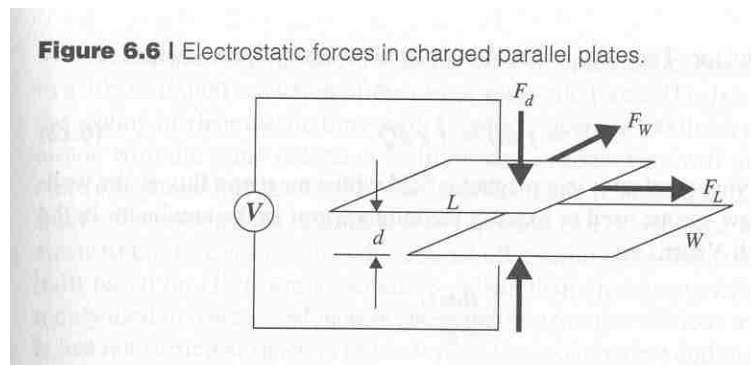
- In Fig. 6.6, the electrostatic forces are,

$$F_d = -\frac{\partial U}{\partial d} = -\frac{1}{2} \frac{\epsilon_r \epsilon_0 W L V^2}{d^2} \quad (2.8)$$

$$F_w = -\frac{\partial U}{\partial W} = \frac{1}{2} \frac{\epsilon_r \epsilon_0 L V^2}{d} \quad (2.10)$$

$$F_L = -\frac{\partial U}{\partial L} = \frac{1}{2} \frac{\epsilon_r \epsilon_0 W V^2}{d} \quad (2.11)$$

- $F_d$ ,  $F_w$ , and  $F_L \propto (l^2)$
- A 10 times reduction in the plate sizes means a 100 times decrease in the induced electrostatic forces.



### 6.5 Scaling in Electromagnetic Forces

- In this section, it is shown that electromagnetic actuation is not scaled down

nearly as favorably as electrostatic forces.

- The **electromagnetic forces** can be induced in a conductor or a conducting loop in a **magnetic field  $\mathbf{B}$**  by **passing current  $i$**  in the conductor.
- The electromotive force (emf) is the force that drives the electrons through the conductor.

- $$U = \frac{1}{2} \frac{\phi^2}{L} \quad (6.13)$$

where  $\phi$  is the magnetic flux, and  $L$  is the inductance.

- Since  $\phi = Li$ , 
$$U = \frac{1}{2} Li^2 \quad (6.14)$$

- The induced electromagnetic force would be

$$F = \left. \frac{\partial U}{\partial x} \right|_{\phi = \text{constant}} \quad (6.15a)$$

$$F = \left. \frac{\partial U}{\partial x} \right|_{i = \text{constant}} \quad (6.15b)$$

- For constant current case,

$$F = \frac{1}{2} i^2 \frac{\partial L}{\partial x}$$

Since  $i \propto l^2$  and  $\partial L / \partial x \propto l^0$ ,  $F \propto l^4$

- If 10 times reduction in size ( $l$ )

$\Rightarrow$  Electromagnetic force: 10,000 times reduction

**Comparison:** Electrostatic force: only 100 times reduction

**Conclusion:** **Electromagnetic force is less favorable in scale-down than Electrostatic force.**

## 6.6 Scaling in Electricity

- Examples: Microsystem actuation by electrostatic, piezoelectric, and thermal resistance heating.

- Electric Resistance: 
$$R = \frac{\rho L}{A} \propto l^{-1} \quad (6.18)$$

where  $\rho$ ,  $L$ , and  $A$  are the resistivity, length, and cross-sectional area, respectively.

- Electric Power Loss: 
$$P = \frac{V^2}{R} \propto l^1 \quad (6.19)$$

where  $V$  is the applied voltage  $\propto l^0$

- Electric field energy density:  $u = \frac{1}{2} \epsilon E^2 \propto l^{-2}$  (6.20)

where the dielectric permittivity  $\epsilon \propto l^0$ , and the electric field  $E \propto l^{-1}$ .

- Example: For a system that carries its own power, the available power  $E_{av} \propto l^3$ .  
 $\Rightarrow \frac{P}{E_{av}} \propto l^{-2}$ . (6.21)

- That is, a **10 times reduction of  $l$**  leads to **100 times greater power loss** due to the resistance increase.
- **Disadvantage** of scaling down of power supply systems.

## 6.7 Scaling in Fluid Mechanics

- In Fig. 6.7, moving the top plate to the right induces the motion of the fluid.

- Newtonian flow:  $\tau \propto \frac{d\theta}{dt}$ , or  $\tau = \mu \frac{d\theta}{dt} = \mu \frac{dV}{dy}$

where  $\tau$ : shear stress;  $\mu$ : coefficient of viscosity (黏滯性);  $d\theta/dt$ : strain rate;  $V$ : fluid velocity.

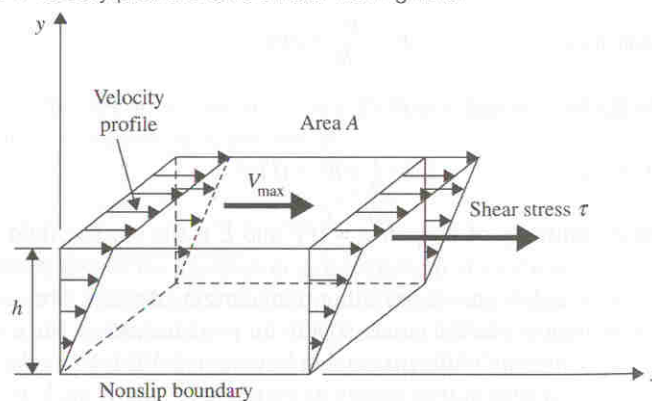
- Thus,  $\mu = \frac{\tau}{R_s}$  (6.22)

where  $R_s = V_{max}/h$

- Rate of volumetric fluid flow:  $Q = A_s V_{ave}$  (6.23)

where  $A_s$ : cross-sectional area for the flow;  $V_{ave}$ : average velocity of the fluid.

**Figure 6.7** | Velocity profile of a volume of moving fluid.



- Reynolds number:  $Re = \frac{\rho VL}{\mu}$

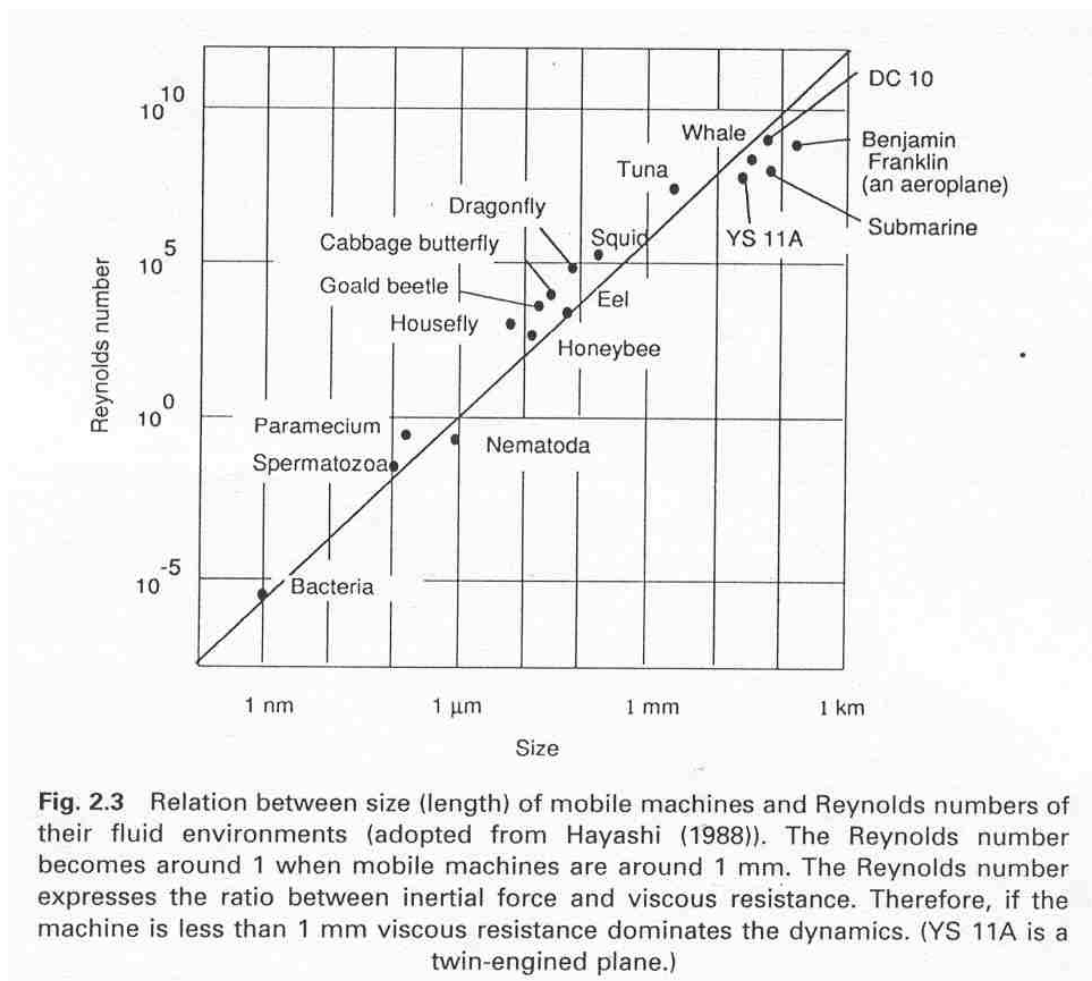
where  $\rho$ : fluid density;  $V$  &  $L$ : characteristic velocity and length scales of the flow.

- $Re \propto$  (inertial forces)/(viscous force)
- **Macro flows**: high inertial forces  $\rightarrow$  high  $Re \rightarrow$  turbulence flow
- **Micro flows**: high viscosity  $\rightarrow$  low  $Re \rightarrow$  laminar flow

p.s.: (1) turbulence flow: fluctuating and agitated;

(2) laminar flow: smooth and steady;

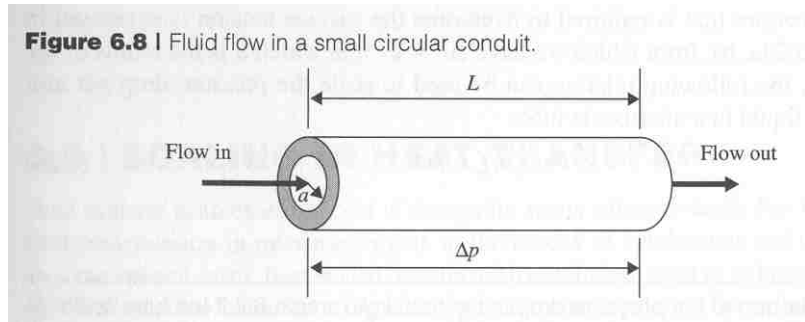
(3) transition from laminar to turbulent:  $10^3 \sim 10^5$



(from “Micromachines: A New Era in Mechanical Engineering,” by Iwao Fujimasa, Oxford University Press, 1996)

- In Fig. 6.8, with the pressure drop  $\Delta P$  over the length  $L$ , the rate of volumetric flow of the fluid is (Hagen-Poiseuille law),

$$Q = \frac{\pi a^4 \Delta P}{8 \mu L} \quad (6.24)$$



- With  $V_{ave} = \frac{Q}{\pi a^2}$ ,  $\Delta P = \frac{8\mu V_{ave} L}{a^2}$  (6.25)

- Thus,  $Q \propto a^4$ ;  $\frac{\Delta P}{L} \propto a^{-2}$

## 6.8 Scaling in Heat Transfer

### 6.8.1 Scaling in Heat Conduction (傳導)

#### Scaling of Heat Flux

- Heat conduction in solid is governed by the Fourier law,

➤  $q_x = -k \frac{\partial T(x, y, z, t)}{\partial x}$

where  $q_x$ : heat flux along the x axis;  $k$ : thermal conductivity of the solid;  $T(x, y, z, t)$ : temperature field.

➤ Rate of heat conduction:  $Q = qA = -kA \frac{\Delta T}{\Delta x}$  (6.27)

- For solids in meso- and microscales,

$$Q \propto (l^2)(l^{-1}) = l^1$$

That is, reduction in size leads to the decrease of total heat flow.

#### Scaling in Submicrometer Regime

- In the submicrometer regime, the thermal conductivity is,

$$k = \frac{1}{3} cV\lambda \propto l^1 \quad (6.28)$$

where  $c$ ,  $V$ , and  $\lambda$  are specific heat, molecular velocity, and average mean free path, respectively.

- Thus,  $Q \propto (l^1)(l^1) = l^2$  (6.29)
- A reduction in size of 10 would lead to a reduction of total heat flow by 100.

### Scaling in Effect of Heat Conduction in Solids of Meso- and Micro-scales

- A dimensionless number, called **the Fourier number,  $F_0$**  is used to determine the time increments in a transient heat conduction analysis.

$$- F_0 = \frac{\alpha t}{L^2}$$

where  $\alpha$ : thermal diffusivity of the material, and  $t$ : time for heat to flow across the characteristic length  $L$ .

$$- t = \frac{F_0}{\alpha} L^2 \propto l^2$$

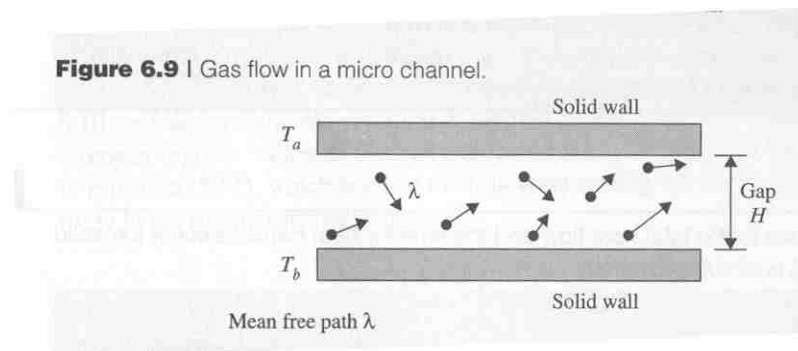
### 6.8.2 Scaling in Heat Convection (對流)

- Heat transfer in fluid is in the mode of convection (Newton's cooling law),

$$Q = qA = hA\Delta T \quad (6.32)$$

where  $Q$ : total heat flow between two plates;  $q$ : heat flux;  $A$ : cross-sectional area for the heat flow;  $h$ : heat transfer coefficient;  $\Delta T$ : temperature difference between these two points.

- $h$ : depends primarily on the fluid velocity, which does not play a significant role in the scaling of the heat flow.
  - Thus, in **meso-** and **micro-**regimes,  $Q \propto A \propto l^2$
- For the cases in which **gases** pass in narrow **channels at submicro-meter scale**,
    - The **classical heat transfer theories** based on continuum fluids **break down**.
    - The seemingly **convective** heat transfer has in fact **become conduction** of heat among the gas molecules as the effect of the boundary layer becomes a dominant factor.
    - In Fig. 6.9,  $H < 7\lambda$   
where  $\lambda = 65\text{nm}$  for gases, and  $1.3 \mu\text{m}$  for liquids.



➤  $\lambda \propto \frac{1}{\rho}$

➤  $k = \frac{1}{3} cV\lambda$

➤  $V = \sqrt{\frac{8kT}{\pi m}}$

where  $T$ : mean temperature of the gas; and  $m$ : molecular weight of the gas.

➤ Effective heat flux:  $q_{eff} = \frac{k\Delta T}{H + 2\varepsilon}$  (6.34)

where  $\Delta T$ : temperature difference between two plates;  $\varepsilon$ : depends on the gases entrapped between two plates,  $2.4 \lambda < \varepsilon < 2.9 \lambda$  for air, O<sub>2</sub>, N<sub>2</sub>, CO<sub>2</sub>, methane, and He, and  $\varepsilon = 11.7 \lambda$  with  $H > 7 \lambda$  for H<sub>2</sub>.